Q: Can we beat the shot-noise limit?

C) Interferometry with coherent + squeezed states

- Squeezed state:

Just like coherent states, squeezed states are also minimum uncertainty states, which can be defined using a scaling transf.

$$\hat{S}(r) = \exp\left[(r^* \hat{a}^2 - r\hat{a}^{\dagger})/2\right], \quad r = |r|e^{i\theta}$$

squeezing factor, phase

$$|\alpha, r\rangle = \hat{S}(r)|\alpha\rangle$$

- For r real, the uncertainties in the quadratures become $(\theta = 0)$

$$\Delta q = e^{-r}/\sqrt{2}, \quad \Delta p = e^{r}/\sqrt{2}$$

$$\Delta q \cdot \Delta p = 1/2.$$

- Squeezed vacuum state

$$|r\rangle = \hat{S}(r)|0\rangle, \quad \langle\hat{N}\rangle = \sinh^2 r$$

- Action of scaling transformation on annihilation and creation operators

$$\begin{cases} \hat{S}(r)\,\hat{a}\,\hat{S}^{\dagger}(r) = \hat{a}\cosh r - \hat{a}^{\dagger}\sinh r \\ \hat{S}^{\dagger}(r)\,\hat{a}\,\hat{S}(r) = \hat{a}^{\dagger}\cosh - \hat{a}\sinh r \end{cases}$$

- Our input state $\quad |\psi\rangle_{in} = |\alpha\rangle_a |r\rangle_b \quad$ Caves, 1981

LIGO

- To calculate the uncertainty $\Delta\varphi$, we need

$$\langle\hat{J}_z\rangle_{out} = \cos\varphi\,\langle\hat{J}_z\rangle_{in} - \sin\varphi\,\langle\hat{J}_x\rangle_{in}$$

$$\Delta^2\hat{J}_z|_{out} = \cos^2\varphi\,\Delta^2\hat{J}_z + \sin^2\varphi\,\Delta^2\hat{J}_x - 2\sin\varphi\cos\varphi\,\mathrm{cov}(\hat{J}_z, \hat{J}_x)$$

$$\hat{J}_z = \frac{1}{2}\left(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}\right)$$

$$\langle \hat{J}_z \rangle_{in} = \frac{1}{2}|\alpha|^2 - \frac{1}{2}\sinh^2 r$$

$$\Delta^2 \hat{J}_z|_{in} = \frac{1}{4}\left(|\alpha|^2 + \frac{1}{2}\sinh^2 2r\right)$$

$$\Delta^2 \hat{J}_x|_{in} = \frac{1}{4}\left(|\alpha|^2 \cosh 2r \underbrace{- \mathrm{Re}(\alpha^2)\sinh 2r} + \sinh^2 r\right), \quad \alpha = \mathrm{Re}(\alpha)$$

$\uparrow$ maximizes

$$\Delta\varphi = \frac{\Delta J_z|_{out}}{\left|\frac{d\langle J_z\rangle_{out}}{d\varphi}\right|} =$$

$$= \left(\frac{\cot^2\varphi\left(|\alpha|^2 + \frac{1}{2}\sinh^2 2r\right) + |\alpha|^2 e^{-2r} + \sinh^2 r}{\left||\alpha|^2 - \sinh^2 r\right|}\right)^{1/2}$$

Optimal operating points $\cot\varphi = 0 \Rightarrow \varphi = \frac{\pi}{2}, \frac{3\pi}{2}$

$$\Delta\varphi = \sqrt{\frac{|\alpha|^2 e^{-2r} + \sinh^2 r}{\left||\alpha|^2 - \sinh^2 r\right|}}$$

- Fix $\langle \hat{N} \rangle$
- $\langle \hat{N} \rangle \gg 1$, vacuum sq. state carrying $\frac{\sqrt{\langle \hat{N} \rangle}}{2}$

$$\langle \hat{N}_s \rangle = \sinh^2 r \approx \frac{1}{4}e^{2r} \approx \frac{\sqrt{\langle \hat{N} \rangle}}{2}$$

- coherent state carries $|\alpha|^2 = \langle \hat{N} \rangle - \frac{\sqrt{\langle \hat{N} \rangle}}{2}$

$$\Delta\varphi \approx \sqrt{\frac{\left(\sqrt{\langle \hat{N} \rangle}\right)/2 - 1/4 + \sqrt{\langle \hat{N} \rangle}/2}{\langle \hat{N} \rangle \left(1 - 1/\sqrt{\langle \hat{N} \rangle}\right)}} \approx \frac{1}{\langle \hat{N} \rangle^{3/4}}$$

Quantum Fisher information for pure states

$$F_Q(\rho_\varphi) = \mathrm{Tr}[\rho_\varphi L_\varphi^2]$$

$$\Rightarrow \Delta\varphi \geq \frac{1}{\sqrt{\nu}\sqrt{F_Q}}$$

n. repetition   QCRB

SLD: $\frac{\partial \rho_\varphi}{\partial \varphi} = \frac{1}{2}(\rho_\varphi L_\varphi + L_\varphi \rho_\varphi)$

- $\rho_\varphi = |\psi_\varphi\rangle\langle\psi_\varphi|$, $L_\varphi = ?$

$$|\partial_\varphi \psi_\varphi\rangle = \frac{\partial |\psi_\varphi\rangle}{\partial\varphi}$$

To get L_φ

$$|\partial_\varphi \psi_\varphi\rangle\langle\psi_\varphi| + |\psi_\varphi\rangle\langle\partial_\varphi\psi_\varphi| = \frac{1}{2}(\rho_\varphi L_\varphi + L_\varphi\rho_\varphi)$$

$$\Rightarrow \boxed{L_\varphi = 2(|\partial_\varphi\psi_\varphi\rangle\langle\psi_\varphi| + |\psi_\varphi\rangle\langle\partial_\varphi\psi_\varphi|)}$$

$$\Rightarrow \langle\psi_\varphi|\psi_\varphi\rangle = 1$$

$$\langle\partial_\varphi\psi_\varphi|\psi_\varphi\rangle + \langle\psi_\varphi|\partial_\varphi\psi_\varphi\rangle = 0$$

$$\langle\partial_\varphi\psi_\varphi|\psi_\varphi\rangle = -\langle\psi_\varphi|\partial_\varphi\psi_\varphi\rangle$$

- $F_Q(|\psi_\varphi\rangle) = \mathrm{Tr}\{|\psi_\varphi\rangle\langle\psi_\varphi| L_\varphi^2\}$

$$\Rightarrow F_Q(|\psi_\varphi\rangle) = 4\left(\langle\partial_\varphi\psi_\varphi|\partial_\varphi\psi_\varphi\rangle - |\langle\partial_\varphi\psi_\varphi|\psi_\varphi\rangle|^2\right)$$

- Unitary imprinting $|\psi_\varphi\rangle = U_\varphi|\psi_0\rangle$

$$U_\varphi = e^{-i\varphi\hat{H}}$$

$$|\partial_\varphi\psi_\varphi\rangle = \frac{dU_\varphi}{d\varphi}|\psi_0\rangle = -i\hat{H}U_\varphi|\psi_0\rangle$$

$$F_Q = 4\left(\langle\psi_0|\frac{dU_\varphi^\dagger}{d\varphi}\frac{dU_\varphi}{d\varphi}|\psi_0\rangle - \left|\langle\psi_0|\frac{dU_\varphi^\dagger}{d\varphi}U_\varphi|\psi_0\rangle\right|^2\right)$$

$$F_Q = 4\left(\langle\Psi_0|(+i\hat{H}U_\varphi^\dagger)(-i\hat{H}U_\varphi)|\Psi_0\rangle - \left|\langle\Psi_0|(+i\hat{H}U_\varphi^\dagger U_\varphi)|\Psi_0\rangle\right|^2\right.$$

$$\Rightarrow F_Q(|\Psi_\varphi\rangle) = 4\left(\langle\Psi_0|\hat{H}^2|\Psi_0\rangle - |\langle\Psi_0|\hat{H}|\Psi_0\rangle|^2\right)$$

$$= 4\,\Delta^2\hat{H}_0^2$$

$$\hat{H} = i\frac{\partial U_\varphi}{\partial\varphi}U_\varphi^\dagger \rightsquigarrow U_\varphi = e^{-i\varphi\hat{H}}$$

$$\Rightarrow |\Psi_\varphi\rangle = |\Psi\rangle_{out} = \underbrace{e^{-i\frac{\pi}{2}\hat{J}_x}e^{-i\varphi\hat{J}_z}e^{i\frac{\pi}{2}\hat{J}_x}}_{U_\varphi}|\Psi\rangle_{in}$$

$$\Rightarrow \hat{H} = \hat{J}_z$$

QFI: $F_Q = 4\left({}_{in}\langle\Psi|\hat{J}_z^2|\Psi\rangle_{in} - |{}_{in}\langle\Psi|\hat{J}_z|\Psi\rangle_{in}|^2\right) = \Delta^2\hat{J}_z|_{in}$

1) Coherent example $\quad |\Psi\rangle_{in} = |\alpha\rangle_a|0\rangle_b$

$$\Delta^2\hat{J}_z|_{in} = 1/4\,|\alpha|^2 = 1/4\,\langle\hat{N}\rangle$$

$$F_Q^{|\alpha\rangle|0\rangle} = \langle\hat{N}\rangle \Rightarrow \boxed{\Delta\varphi \geq \frac{1}{\sqrt{\langle\hat{N}\rangle}}}$$

2) Fock state $\quad |\Psi\rangle_{in} = |N\rangle_a|0\rangle_b$

$$\Delta^2\hat{J}_z|_{in} = 1/4\langle\hat{N}\rangle = 1/4\,N$$

QFI: $F_Q^{|N\rangle|0\rangle} = N \Rightarrow \boxed{\Delta\varphi \geq \frac{1}{\sqrt{N}}}$

Entanglement

Separable state $\quad \rho_{AB} = \sum_i p_i \rho_i^{(A)} \otimes \rho_i^{(B)}$, $p_i \geq 0$

Entangled states can be defined as all states that are not separable.

$|\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_N\rangle$ ⇝ pure separable states

Example: two qubits

$$|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_A |1\rangle_B \pm |1\rangle_A |0\rangle_B \right)$$

- $|NOON\rangle = \frac{1}{\sqrt{2}} \left(|N\rangle_a |0\rangle_b + |0\rangle_a |N\rangle_b \right) = |\psi_0\rangle$

- $U_\varphi = e^{i\varphi \hat{a}^\dagger \hat{a}} \otimes \mathbb{1}$

$$|\psi_\varphi\rangle = e^{i\varphi \hat{a}^\dagger \hat{a}} \otimes \mathbb{1} \, |NOON\rangle = \frac{1}{\sqrt{2}} \left(e^{iN\varphi} |N,0\rangle + |0,N\rangle \right)$$

$$|\partial_\varphi \psi_\varphi\rangle = \frac{1}{\sqrt{2}} \left(iNe^{iN\varphi} |N,0\rangle \right)$$

$$\langle \partial_\varphi \psi_\varphi | \partial_\varphi \psi_\varphi \rangle = \frac{1}{2} (-iN)(iN) = \frac{N^2}{2}$$

$$\langle \partial_\varphi \psi_\varphi | \psi_\varphi \rangle = \frac{1}{2} iN$$

$F_Q^{|NOON\rangle} = N^2 \quad \Rightarrow \quad \Delta\varphi \geq \frac{1}{N}$ ⇝ Heisenberg limit