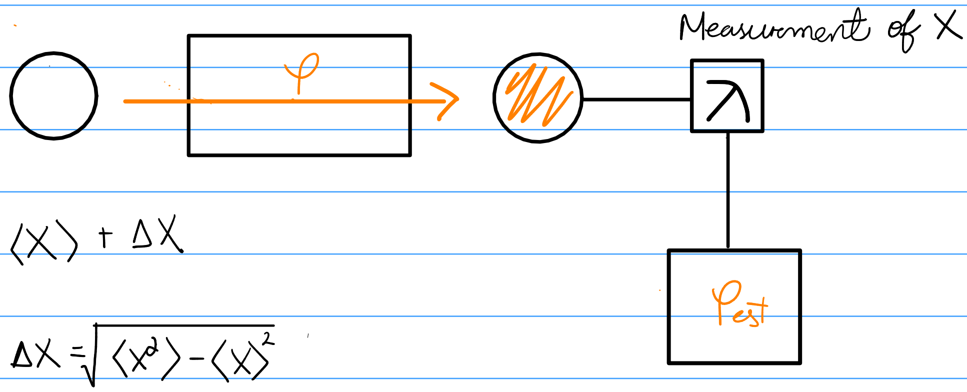
Metrology: measuring a system and learning unknown parameter



$\langle X \rangle \pm \Delta X$

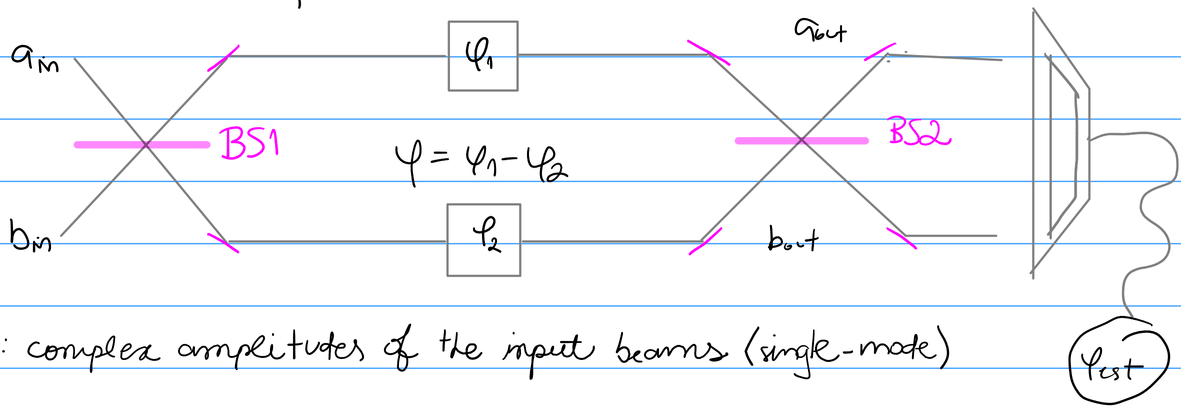
$$\Delta X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2}$$

$$\Delta\varphi = \sqrt{\langle \varphi_{est}^2 \rangle - \langle \varphi_{est} \rangle^2} \sim \text{estimation uncertainty}$$

- Uncertainty propagation formula: $\Delta\varphi = \dfrac{\Delta X}{\left|\dfrac{d\langle X\rangle}{d\varphi}\right|}$

- OUR FOCUS: Optical interferometry (Progress in Optics 60, 345 (2015))

Mach-Zehnder interferometer:



a, b : complex amplitudes of the input beams (single-mode)

* High phase sensitivity is important in many applications
e.g. LIGO interferometer

* BS1 : 50/50; the beams acquire $-\pi/2$ phase

$$\begin{pmatrix} a_{in} \\ b_{in} \end{pmatrix} \Rightarrow \quad BS1: \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & e^{-i\pi/2} \\ e^{-i\frac{\pi}{2}} & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

e.g. action of beam splitter on beam a

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} a_{in} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} a_{in} \\ -i a_{in} \end{pmatrix}$$

- BS2: beams acquire a $\pi/2$ phase

- Total transformation of the input beams

$$\begin{pmatrix} a'_{out} \\ b'_{out} \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}}_{BS2}\begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix}\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}}_{BS1}\begin{pmatrix} a_{in} \\ b_{in} \end{pmatrix}$$

* Nonclassical features of light: better sensitivity?
  - Available light sources: single-photon, coherent and squeezed states.

- To deal with this, we replace the complex field amplitude $\hat{a}$ with the operator $\hat{a}$:

$$a \to \hat{a} \qquad a^* \to a^\dagger \qquad [\hat{a}, \hat{a}^\dagger] = 1$$
$$b \to \hat{b} \qquad b^* \to b^\dagger$$

- A single-mode can contain a large number of photons (photons are bosons)
  - $\rightsquigarrow$ state with definite number of photons: $|N\rangle$,

$$\begin{cases} \hat{a}|N\rangle = \sqrt{N}\,|N-1\rangle \\ \hat{a}^\dagger|N\rangle = \sqrt{N+1}\,|N+1\rangle \end{cases}$$

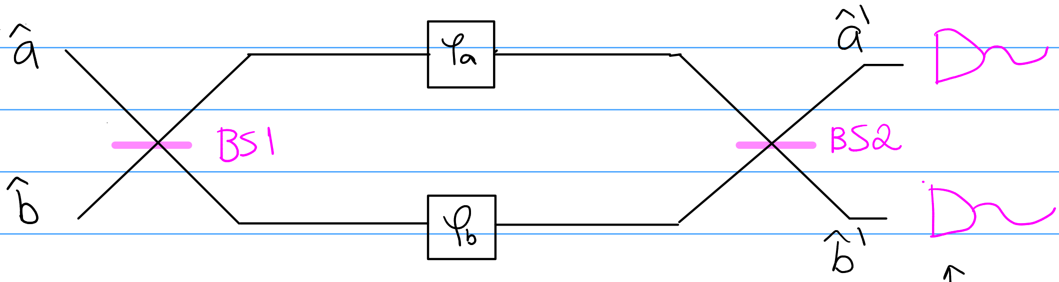
$\rightsquigarrow$ $\hat{a}^\dagger\hat{a}|N\rangle = N|N\rangle \to \hat{N} = \hat{a}^\dagger a \to$ Hermitian

$\rightsquigarrow$ $\{|N\rangle\}_N \to$ Fock basis

$\rightsquigarrow$ Pure state $\qquad |\psi\rangle = \sum_N c_N |N\rangle$

$\rightsquigarrow$ Born rule: $\qquad P(N) = |\langle N|\psi\rangle|^2 = |c_N|^2$

... Back to the MZ interferometer:



$\boxed{\varphi = \varphi_a - \varphi_b}$, $\hat{n} = \hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}$

$$\begin{pmatrix}\hat{a}' \\ \hat{b}'\end{pmatrix} = \frac{1}{2}\underbrace{\begin{pmatrix}1 & i \\ i & 1\end{pmatrix}}\underbrace{\begin{pmatrix}e^{i\varphi_a} & 0 \\ 0 & e^{i\varphi_b}\end{pmatrix}}\underbrace{\begin{pmatrix}1 & -i \\ -i & 1\end{pmatrix}}\begin{pmatrix}\hat{a} \\ \hat{b}\end{pmatrix}$$

$$= \underline{e^{i(\varphi_a+\varphi_b)}}\begin{pmatrix}\cos(\varphi/2) & -\sin(\varphi/2) \\ \sin(\varphi/2) & \cos(\varphi/2)\end{pmatrix}\begin{pmatrix}\hat{a} \\ \hat{b}\end{pmatrix}$$

- Jordan-Schwinger transformation (Yurke et al., 1986)

$$\hat{J}_x = \frac{1}{2}(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a})$$

$$\hat{J}_y = \frac{i}{2}(\hat{b}^\dagger\hat{a} - \hat{a}^\dagger\hat{b}) \quad \Rightarrow \quad [\hat{J}_x, \hat{J}_y] = iJ_z$$

$$\underline{\hat{J}_z = \frac{1}{2}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})} = \frac{1}{2}\hat{n}$$

Total angular momentum $\hat{J} = \frac{\hat{N}}{2}\left(\hat{N}+1\right)$, $\hat{N} = \hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}$

→ Action of elements

$$\hat{a}' = U\hat{a}U^\dagger, \quad \hat{b}' = U\hat{b}U^\dagger, \quad U = \exp(-i\alpha\hat{J}\cdot\hat{s})$$

→ Balanced beam splitter: $U = e^{-i\pi/2\,\hat{J}_x}$

→ Phase delay: $U = e^{-i\varphi\hat{J}_z}$

We can now write

$$\begin{pmatrix} \hat{J}_x' \\ \hat{J}_y' \\ \hat{J}_z' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}_{\text{rotation x-axis}} \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation y-axis}} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{\text{rotation x}} \begin{pmatrix} \hat{J}_x \\ \hat{J}_y \\ \hat{J}_z \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix} \begin{pmatrix} \hat{J}_x \\ \hat{J}_y \\ \hat{J}_z \end{pmatrix}$$

Therefore, we get

$$J_x' = \cos\varphi\, \hat{J}_x + \sin\varphi\, \hat{J}_z$$
$$\hat{J}_y' = \hat{J}_y$$
$$\hat{J}_z' = \cos\varphi\, \hat{J}_z - \sin\varphi\, \hat{J}_x$$

$\rightsquigarrow$ Heisenberg picture.

Note that

$$|\psi\rangle_{out} = \exp(-i\pi/2\, \hat{J}_x)\, \exp(-i\varphi \hat{J}_z)\, \exp(i\pi/2\, \hat{J}_x)\, |\psi_{in}\rangle$$

From the Heisenberg picture,

$$\boxed{\langle \hat{J}_z \rangle_{out} = \cos\varphi \langle \hat{J}_z \rangle_{in} - \sin\varphi \langle \hat{J}_x \rangle_{in}}$$

and the variance reads

$$\boxed{\Delta^2 J_z|_{out} = \cos^2\varphi\, \Delta^2 \hat{J}_z|_{in} + \sin^2\varphi\, \Delta^2 J_x|_{in} - 2\sin\varphi\cos\varphi\, \mathrm{cov}(\hat{J}_x, \hat{J}_z)|_{in}}$$

where $\mathrm{cov}(\hat{J}_x, \hat{J}_z) = 1/2 \langle \hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x \rangle - \langle \hat{J}_x \rangle \langle \hat{J}_y \rangle$

- Precision of estimation $\Delta\varphi$:

$$\boxed{\Delta\varphi = \frac{\Delta \hat{J}_z|_{out}}{\left|\frac{d\langle \hat{J}_z \rangle_{out}}{d\varphi}\right|}}$$

A) Fock state interferometry

$$\boxed{|\psi\rangle_{in} = |N\rangle_a |0\rangle_b}$$

$$\Rightarrow \hat{J}_z |N\rangle_a |0\rangle_b = \frac{1}{2}(\hat{a}^+\hat{a} - \hat{b}^+\hat{b}) |N\rangle_a |0\rangle_b = \frac{1}{2} N |N\rangle_a |0\rangle_b$$

$$\Delta^2 \hat{J}_z |_{in} = 0$$

$$\langle \hat{J}_z \rangle_{in} = \langle N, 0 | \hat{J}_z | N, 0 \rangle = \frac{1}{2} N$$

$$\langle \hat{J}_x \rangle_{in} = \frac{1}{2} \langle N, 0 | \hat{a}^+\hat{b} + \hat{b}^+\hat{a} | N, 0 \rangle = 0$$

$$\langle 0 | \hat{b} | 0 \rangle = 0$$

$$\hat{b} |0\rangle = 0$$

$$\Delta^2 J_x |_{in} = \langle \hat{J}_x^2 \rangle_{in} - \cancel{\langle \hat{J}_x \rangle}_{in}^2$$

$$\hat{J}_x^2 = \frac{1}{4}(\hat{a}^+\hat{a}\, \hat{b}\hat{b}^+ + \hat{a}^+\hat{a}^+ \hat{b}\hat{b} + \hat{a}\hat{a}^+ \hat{b}^+\hat{b} + \hat{a}\hat{a}\, \hat{b}^+\hat{b}^+)$$

$$\langle \hat{J}_x^2 \rangle_{in} = \frac{1}{4} N$$

$$\Delta^2 J_z |_{out} = \frac{1}{4} N \sin^2\varphi$$

$$\langle \hat{J}_z \rangle_{out} = \frac{1}{2} N \cos\varphi$$

$$\rightarrow \Delta\varphi = \frac{\Delta \hat{J}_z |_{out}}{\left| \frac{d \langle \hat{J}_z \rangle_{out}}{d\varphi} \right|} = \frac{\frac{1}{2}\sqrt{N}\, |\cancel{\sin\varphi}|}{N\, |\cancel{\sin\varphi}|/2} = \frac{1}{\sqrt{N}}$$

shot-noise limit.

B) Coherent states

$$|\psi_{in}\rangle = |\alpha\rangle_a |0\rangle_b$$

→ $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, and thus $\langle\alpha|\hat{a}^+ = \alpha^* \langle\alpha|$, $\alpha = |\alpha|e^{i\theta}$

→ average number of photons : $\langle\alpha|\underset{\hat{a}^+\hat{a}}{\hat{N}}|\alpha\rangle = \alpha^*\alpha = |\alpha|^2$

→ Fock basis

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$P(n) = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{2n!} \quad \leftarrow \text{Poisson dist.}$$

$$\boxed{\begin{gathered} \hat{q}_\theta = \frac{1}{\sqrt{2}}(\hat{a}e^{-i\theta} + \hat{a}^+e^{i\theta}), \quad \hat{p}_\theta = q_{\theta+\frac{\pi}{2}} = \frac{-i}{\sqrt{2}}(\hat{a}e^{-i\theta} - \hat{a}^+e^{i\theta}) \\ [\hat{q}_\theta, \hat{p}_\theta] = i \quad \Rightarrow \quad \Delta p_\theta \cdot \Delta q_\theta \geq 1/2 \end{gathered}}$$

→ in a coherent state $|\alpha\rangle$

$$\Delta p_\theta \cdot \Delta q_\theta = 1/2.$$

→ $\langle\hat{J}_z\rangle_{in} = \frac{1}{2}\langle\alpha, 0 | \hat{a}^+\hat{a} - \hat{b}^+\hat{b} |\alpha, 0\rangle = \frac{1}{2}|\alpha|^2$

→ $\langle\hat{J}_x\rangle_{in} = 0$

→ $\Delta^2 \hat{J}_z|_{in} = \Delta^2 \hat{J}_x|_{in} = \frac{1}{4}|\alpha|^2$

$$\Delta\varphi^{|\alpha\rangle_a|0\rangle_b} = \frac{\Delta\hat{J}_z|_{out}}{\left|\frac{d\langle\hat{J}_z\rangle_{out}}{d\varphi}\right|} = \frac{\frac{1}{2}|\alpha|}{\frac{1}{2}|\alpha|^2|\sin\varphi|} = \frac{1}{|\alpha||\sin\varphi|} = \frac{1}{\sqrt{\langle\hat{N}\rangle}|\sin\varphi|} = \frac{1}{\sqrt{\langle\hat{N}\rangle}}$$

- optimal operating points $\varphi = \frac{\pi}{2}, \frac{3\pi}{2} \ldots$