A very quick intro :

1 A closer look at quantum enhanced metrology :

1a Ramsey interferometer
1b quantum - enhanced interferometer

2 Multiparameter Quantum Metrology

2a Classical multiparameter metrology
2b SLD - QCR bound
2c Holevo Cramer Rao bound

3 Quantum Local Asymptotic Normality

4 Quantum - enhanced Multiparameter metrology

Some References

- chapter 1

1) Review on enhanced quantum metrology : https://arxiv.org/abs/1102.2318

- chapter 2

1) Perspective on multiparameter metrology → https://arxiv.org/abs/1911.12067
2) Review with emphasis on multiparameter metrology → https://arxiv.org/abs/1604.02615

- chapter 3

1) Chapter 4 of → https://arxiv.org/abs/2001.11742
Reference is useful also for general review of multiparameter metrology

- chapter 4

1) Article → https://arxiv.org/abs/1307.7653

① Enhanced Quantum Metrology

I would like to start from what Paulo explained last time, so I will recall a few results that will be useful:

- QFI for unitary encoding $|\Psi_\theta\rangle = \exp\{i\theta G\}|\Psi_0\rangle \Rightarrow$

$$F_Q = 4\left\{\langle\Psi_0|G^2|\Psi_0\rangle - \langle\Psi_0|G|\Psi_0\rangle^2\right\}$$

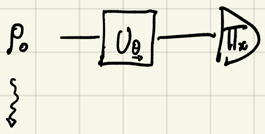
Then Paulo showed that for the estimation of a phase encoded via operator $\rightsquigarrow$ $U_\varphi = \exp\{i\varphi a^\dagger a\}$ and with input state

$$|NOON\rangle = \frac{1}{\sqrt{2}}\left\{|N\rangle_a|0\rangle_b + |0\rangle_a|N\rangle_b\right\} \rightsquigarrow F_Q = N^2 \Rightarrow \Delta\varphi \geq \frac{1}{\sqrt{\nu}\,N} \longrightarrow$$

we have ν inputs

$\to$ compared to coherent state $|\alpha\rangle$ with same number of photons $\rightsquigarrow$ $\Delta\varphi \geq \frac{1}{\sqrt{\nu N}} \Rightarrow$ advantage of $\frac{1}{\sqrt{N}}$!

Here the resource is the number of "photons" N that enhance the scaling $\rightsquigarrow$ we are optimizing on a single probe



(*)

we are basically optimizing over this $\rho_0 \sim |NOON\rangle$ states etc.

I will consider a more information theoretic approach where the resource is the number of probes / the number of samples

To do so, and to exploit quantum effects, we need to generalize the situation in (*).

(12) Ramsey Interferometer

Let us consider the case in which we have to estimate a phase from a unitary

$$U_\theta = \exp\{-i\,\theta\,\sigma_z\} \qquad \sigma_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow$$

→ what is the optimal state ? →

- One can show that pure state are always better than mixed states
- For unitary encoding → G has eigenvalues and eigenvectors

$$\{\lambda_i, |\psi_i\rangle\} \longrightarrow \text{optimal state} \quad |\psi_{opt}\rangle = \frac{|\lambda_{max}\rangle + |\lambda_{min}\rangle}{\sqrt{2}} \rightarrow$$

imprints the largest phase

In our case the generator is σ_z ⇒ optimal state is ⟶

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left\{|0\rangle + |1\rangle\right\}$$

$$|\psi_\theta\rangle = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}\left(e^{-i\frac{\theta}{2}}|0\rangle + e^{i\frac{\theta}{2}}|1\rangle\right) =$$

$$\simeq \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\theta}|1\rangle\right)$$

$$QFI = 4\left\{\langle\psi_0|G^2|\psi_0\rangle - \langle\psi_0|G|\psi_0\rangle^2\right\}$$

$$G = \sigma_z \Rightarrow G^2 = \frac{1}{4}\mathbb{1} \Rightarrow \langle\psi_0|G^2|\psi_0\rangle = \frac{1}{4}$$

$$\hookrightarrow \langle\psi_0|\sigma_z|\psi_0\rangle = \langle\psi_0|\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = 0$$

$$QFI = 4\cdot\frac{1}{4} = 1$$

What is the optimal measurement? We can look at SLD, or have a guess

$$p(+|\theta) = |\langle +|\psi_\theta\rangle|^2 = (1-\cos\theta)/2$$
$$p(-|\theta) = |\langle -|\psi_\theta\rangle|^2 = (1+\cos\theta)/2$$

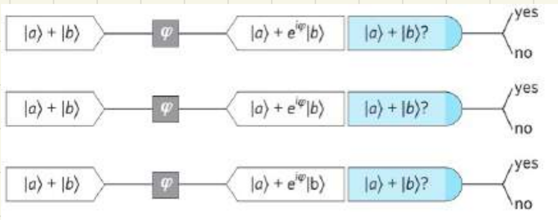
$$FI = \sum_{x=\pm}\left(\frac{\partial}{\partial\theta}\log p(x|\theta)\right)^2 p(x|\theta) \qquad \frac{\partial}{\partial\theta}\log p(x|\theta) = \frac{1}{p(x|\theta)}\partial_\theta p(x|\theta)$$

$$= \sum_{x=\pm}\frac{(\partial_\theta p(x|\theta))^2}{p(x|\theta)} = \sum_{x=\pm}\frac{(\sin\theta)^2}{4\,p(x|\theta)} = \frac{1}{2}\sin^2\theta\left\{\frac{1}{1-\cos\theta}+\frac{1}{1+\cos\theta}\right\}$$

$$= \frac{\cancel{\sin^2\theta}}{2}\left\{\frac{2}{\cancel{1-\cos^2\theta}}\right\} = 1 \quad \#$$

$$\Delta\theta_{cc} \geq \frac{1}{\sqrt{N\,H(\theta)}} = \frac{1}{\sqrt{N}}$$

This is known as a classical - classical strategy → we use each probe individually and we measure them individually



- Can we do better with these resources?

↓

To see that we need to look for alternative strategy than the classical parallel strategy we have studied here.

1b Enhanced Ramsey Interferometer

So far so good, but as I said to get an enhancement we need to consider a more general situation than probing a single system.

As we said at the beginning, a metrological protocol consists of three parts: preparation, interaction, measurement.

If we have $|\psi_0\rangle^{\otimes N}$ initially prepared states $\longrightarrow$ we can "pre-process" them to make more sensible to our interaction, i.e. our encoding of the parameter.

The quantum strategy that we will explore is the following

$$|\psi_0\rangle^{\otimes N} \longmapsto |GHZ_N\rangle = \frac{1}{\sqrt{2}}\left\{|0\rangle^{\otimes N} + |1\rangle^{\otimes N}\right\}$$

Now applying $U_\theta^{\otimes N} \mapsto$

$$U_\theta^{\otimes N}|GHZ_N\rangle = \frac{1}{\sqrt{2}}\left\{|0\rangle^{\otimes N} + e^{i\theta N}|1\rangle^{\otimes N}\right\} = |\psi_{\underset{\rightarrow}{\theta}}^{(N)}\rangle$$

$\Rightarrow$ What is the QFI ? There are different ways to compute it. The most direct way is the following.

$$|0\rangle^{\otimes N} = |0\rangle \qquad |1\rangle^{\otimes N} = |1\rangle \qquad \tilde{\theta} = \theta N \quad \Rightarrow \quad \frac{1}{\sqrt{2}}\left\{|0\rangle + e^{i\tilde{\theta}}|1\rangle\right\} \longrightarrow$$

This is the previous case $\Rightarrow$ $H(\tilde{\theta}) = 1$

However we want to estimate θ, not $\tilde{\theta}$. This correspond to a reparametrization:

$$H(\tilde{\theta}) = H(\theta)\left(\frac{d\theta}{d\tilde{\theta}}\right)^2 \quad \Longrightarrow \quad \text{in our case } \theta = \frac{\tilde{\theta}}{N} \Rightarrow \frac{d\theta}{d\tilde{\theta}} = \frac{1}{N}$$

$$1 = \frac{H(\theta)}{N^2} \quad \Rightarrow \quad H(\theta) = N^2$$

⊛ Exercise : prove it by the standard calculations

One can also show that the optimal measurement is given by the dicotomic measurement

$$\Pi = \{ |GHZ_N \rangle\langle GHZ_N|, \bigotimes_{i=1}^{N} \mathbb{1}_2 - |GHZ_N\rangle\langle GHZ_N| \}$$

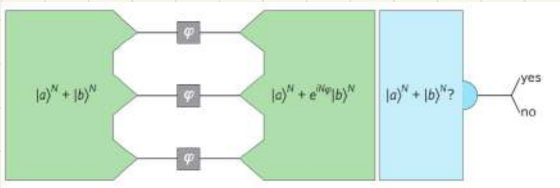
↳ entangled measurement

⊛ Exercise : evaluate the $P_{ent} = |\langle GHZ_N | \Psi_{\vec{\theta}}^{(N)} \rangle|^2$ and the FI of probability $\{P_{ent}, 1 - P_{ent}\}$ and check that corresponds to $H(\theta) = N^2$.

⇓

We have found the measurement strategy to saturate the Quantum bound

This is a Quantum - Quantum strategy



However with this strategy, by using N probes collectively, we obtain a sample of size "1" : we can not saturate the Classical Cramer Rao bound with only one single outcome!

Solution → Split the N probes into groups of n probe such that we have $\nu = \frac{N}{n}$ sample of the entangled measurement

In this case → $H(\theta) = n^2$ and QCR b reads as

$$\Delta \hat{\theta}_Q \geq \frac{1}{\sqrt{\nu \quad n^2}} = \frac{1}{\sqrt{\frac{N}{n} n^2}} = \frac{1}{\sqrt{N \cdot n}}$$

Compared to the scaling of individual probes → $\Delta \theta_c \geq \frac{1}{\sqrt{N}}$

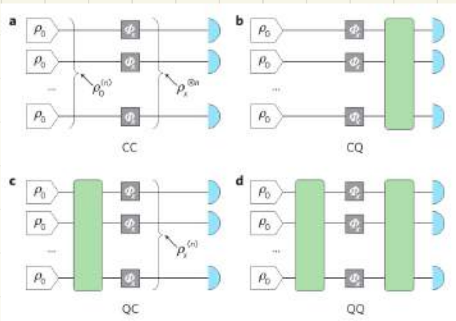
improvement of $\sqrt{n}$ in the scaling! If ν is sufficiently large to have a large sample that saturate the CCR bound.

So let us list the conditions to have a quantum enhancement:

- Be able to prepare entangled state of depth "n"
- Be able to perform an entanglement measure on the entangled state
- Have enough entangled state to saturate the classical Cramer Rao bound, i.e. a large enough sample.

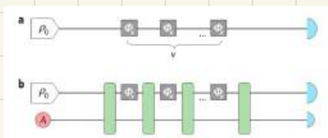
So, just to make an example: if $\nu = 10^4$ to saturate CCR and $N = 10^6 \Rightarrow n = 10^2 \Rightarrow$ if I'm able to prepare an entangled state of such depth $\Rightarrow$ I get an enhancement of a factor 10 by using entangled probes rather than individual probes.

In general $\rightarrow$ 4 different strategies



$\Rightarrow$ for unitary channels QC and QQ can both achieve Heisenberg scaling

- Two final comments:
  1) In general these scenarios are extremely sensible to noise $\longrightarrow$ $\rightarrow$ reduce the enhancement to a costant factor. Solutions to avoid these have been proposed $\rightarrow$ quantum error corrections; continuous monitoring.
  2) Sequential strategies are the most general scenario. No need for entanglement.



② Multiparameter quantum metrology

After having discussed a bit the framework of enhanced quantum metrology, we now move to what is known as multiparameter quantum metrology. But before addressing the quantum version, it is important to look at the classical case with its properties and differences with the single parameter case.

Single parameters are often an oversimplification of real-life metrological setups → examples:

1) Usually one would like to estimate a parameter θ of a statistical model that it is assumed to be true. But in general we have to deal with noise and theoretical model does never match exactly the physical model. To introduce noise → noisy parameters! According to our guess on the noise we might have extra parameters to estimate ⇝ inevitably a multiparameter scenario.

   ↳ • Estimation of phase and loss in optical interferometric experiments
   • " " " " dephasing in atomic interferometry

2) There are specific physical scenario where multiple parameters are natural:

   • Quantum Imaging → reconstruction of an object using the phase acquired upon transmission → each pixel corresponds to a phase ⇒ ⇒ multi parameter

   • Sensing of vectors field (even gravitational waves!)

Let us consider a statistical model that depends from d parameters

$p(x|\vec{\theta})$ with $\vec{\theta} = \{\theta_1, \dots, \theta_d\}$ The idea of multiparameter estimation is that given a <u>single</u> sample $\{x_1, \dots, x_N\}$ we want to simultaneously estimate the parameters $\underset{\to}{\theta}$.

In this case, the figure of merit that measure the precision of the estimators $\{\hat{\theta}_i\}$ is the mean square error matrix

$$MSE[\underset{\rightarrow}{\hat{\theta}}]_{ij} = \mathbb{E}[(\theta_i - \hat{\theta}_i)(\theta_j - \hat{\theta}_j)] \xrightarrow[\text{unbiased estimator}]{} Cov[\underset{\rightarrow}{\hat{\theta}}]_{ij} =$$

$\hookrightarrow \mathbb{E}[\underset{\rightarrow}{\hat{\theta}}] = \underset{\rightarrow}{\theta}$

$$= \mathbb{E}[(\theta_i - \mathbb{E}[\hat{\theta}_i])(\theta_j - \mathbb{E}[\hat{\theta}_j])] \rightsquigarrow \text{covariance matrix.}$$

This matrix satisfy a Matrix version of the Cramer Rao Bound

$$Cov[\underset{\rightarrow}{\hat{\theta}}] \geq \frac{1}{N} \mathcal{F}(\underset{\rightarrow}{\theta})^{-1}$$ where the matrix elements are given as

$$\mathcal{F}(\underset{\rightarrow}{\theta})_{ij} = \sum_x p(x;\underset{\rightarrow}{\theta}) \left(\frac{\partial}{\partial\theta_i} \log p(x;\underset{\rightarrow}{\theta})\right)\left(\frac{\partial}{\partial\theta_j} \log p(x;\underset{\rightarrow}{\theta})\right)$$

$$= \mathbb{E}\left[\left(\frac{\partial}{\partial\theta_i} \log p(x;\underset{\rightarrow}{\theta})\right)\left(\frac{\partial}{\partial\theta_j} \log p(x;\underset{\rightarrow}{\theta})\right)\right]$$

Some properties (distinct from the single-parameter scenario)

- It is a positive semidefinite matrix.
- Diagonal terms $\longrightarrow$ We notice that the ith diagonal element correspond to the classical Fisher Information for the ith parameter θ_i.
- Off diagonal terms: The off diagonal elements instead represent the fact that for this choice of parameters there is a correlation between changing two parameters and how the function change itself.

this is reflected also on the estimator $\hat{\theta}_i$ that will be correlated.

- Orthogonal parameters: For two parameter θ_i and θ_j, $\mathcal{F}(\underset{\rightarrow}{\theta})_{ij} = 0$
$\Rightarrow$ θ_i and θ_j are orthogonal parameters $\rightarrow$ the maximum likelihood estimates are asymptotically uncorrelated (and so error do not propagate $\rightarrow$ the variance of MLE is the same if do not know the other parameters)

If not $\rightarrow$ statistical correlations among the parameters increase the error on the ith parameter if all the other parameters are known

$$\hookrightarrow \quad Cov[\hat{\underset{\rightarrow}{\theta}}] \geq \frac{1}{N} \mathcal{F}(\underset{\rightarrow}{\theta})^{-1} \rightarrow \text{scalar bound}$$

$$\rightarrow \quad tr\{Cov[\hat{\underset{\rightarrow}{\theta}}]\} \geq \frac{1}{N} \sum_x [\mathcal{F}(\underset{\rightarrow}{\theta})^{-1}]_{xx} \geq \frac{1}{N} \sum_x \frac{1}{\mathcal{F}(\theta)_{xx}}$$

saturate if $\mathcal{F}(\underset{\rightarrow}{\theta})$ is diagonal

- Change of parameters:

The fact that we have multiple parameters give us an extra freedom of reparametrization i.e we can reparametrize the $\{\theta_i\}_{i=1}^{d}$ and this will affect the Fisher Information as follows.

$$\underset{\rightarrow}{\theta'} = f(\underset{\rightarrow}{\theta}) \quad \Rightarrow \quad \Rightarrow \quad \mathcal{F}(\underset{\rightarrow}{\theta'}) = M^T \mathcal{F}(\underset{\rightarrow}{\theta}) M \quad \text{with}$$

$$M_{ij} = \frac{\partial \theta_i}{\partial \theta'_j}$$

In particular given the positive semidefiniteness of $\mathcal{F}(\underset{\rightarrow}{\theta})$, it can always be diagonalized. This is in one-to-one correspondence with a reparametrization. This implies that there is always a reparametrization for which the parameter are completely uncorrelated.

- Singular statistical models

On the other hand, it's only positive semidefinite $\Rightarrow$ it might be singular.

If $\mathcal{F}$ is singular $\Rightarrow$ SINGULAR STATISTICAL MODEL :

the simultaneous estimation of the "singular" parameter is impossible, only a function of them (can be found by block-diagonalising)

This is something that one wants to avoid, but sometimes it's a nice property when we want a form of privacy of the information and we want to give access only on a partial information on the parameters, i.e. a function of them.

- Attainable by MLE under regularity conditions

$\Downarrow$

Apart from parametrization and non-orthogonality, the classical multiparameter problem does not introduce any additional difficulties compared to single parameter.

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Let us now move to the Quantum Scenario where $p(x;\theta)$ comes from Born Rule

$\hookrightarrow$ $p(x;\vec{\theta}) = \mathrm{tr}\{\rho_{\vec{\theta}}\, \Pi_x\}$

The classical Cramer Rao bound will now depend on the specific measurement $\{\Pi_x\}_x$. To avoid that and have a general bound that depends only on the quantum statistical model $\rho_{\vec{\theta}}$, more general bounds have been derived in the literature.

In this lecture we will review two of them, given their overall importance in the literature: the SLD - Quantum Cramer Rao bound and the Holevo Cramer Rao bound, respectively SLD - QCR and HCR

- SLD Quantum Cramer Rao bound

The most basic multiparameter bound for quantum statistical models it is basically an extension of the single-parameter quantum Cramer Rao bound.

Indeed, recalling the definition of the SLD as

$$2\,\underset{\substack{\downarrow \\ \partial_{\theta_i}}}{\partial_i}\, \rho_{\underset{\rightarrow}{\theta}} = \underset{\substack{\downarrow \\ L_{\theta_i}}}{L_i}\, \rho_{\underset{\rightarrow}{\theta}} + \rho_{\underset{\rightarrow}{\theta}}\, L_i$$

We have the following matrix inequality

$$\mathrm{Cov}[\hat{\underset{\rightarrow}{\theta}}] \geq \frac{1}{N} F(\underset{\rightarrow}{\theta})^{-1} \geq \frac{1}{N} Q(\underset{\rightarrow}{\theta})^{-1} \longrightarrow \text{SLD - QCR bound}$$

where $Q(\underset{\rightarrow}{\theta})_{ij} = \mathrm{tr}\left\{ \rho_{\underset{\rightarrow}{\theta}} \frac{1}{2}\left\{ L_i L_j + L_j L_i \right\}\right\}$ SLD - Quantum Fisher Information matrix SLD - QFIM

As we notice, for a single parameter, this reduce the single parameter quantum cramer Rao bound, which we know can be saturated, i.e. there exists a POVM whose FI is equal to the QFI.

This is not the case for multiple parameters, and this is the huge difference between single and multiple parameters at the quantum level: in general the SLD - QCR bound can not be saturated, i.e. $\nexists$ a POVM $\{\Pi_x\}$ whose FIM is equal to the SLD - QFIM.

Why? I would like to give an heuristic argument on this.

Let us consider two parameter θ_1 and θ_2 encoded as

$$|\psi_{\underset{\rightarrow}{\theta}}\rangle = e^{\frac{i}{2}(\theta_1\sigma_x - \theta_2\sigma_y)}|0\rangle$$

exercise

One can compute the SLD for θ_1 and θ_2 $\Rightarrow$

$$L_1 = \sigma_y \quad \text{and} \quad L_2 = \sigma_x$$

Each of these identify the optimal measurement that saturate the single parameter QCR bound.

In order to saturate the SLD-QCR bound inequality we need to perform these two optimal measurements simultaneously, and this is only possible if the two optimal measurement commute, which in general is not the case! Here is not the case, $[L_1, L_2] = 0$

There is an extra problem: Given two POVMS $\{\Pi_x^{(1)}\}$ and $\{\Pi_y^{(2)}\}$, it would be nice to know which one perform better to estimate our $\underset{\rightarrow}{\theta}$ parameters.

However, the order in the space of positive semidefinite matrix is only partial, i.e. there may be pairs for which neither elements proceed the other

$\downarrow$

For this reason is necessary to introduce scalar bounds. $\Rightarrow$

$\Rightarrow$ the most natural one is given by the trace of the two sides of the inequality

$\Downarrow$

$$\mathrm{Tr}\{\mathrm{Cov}[\hat{\underset{\rightarrow}{\theta}}]\} \geq C^{SLD}[\underset{\rightarrow}{\theta}] = \frac{1}{N}\mathrm{Tr}\{Q(\underset{\rightarrow}{\theta})^{-1}\} \rightarrow \text{the scalar QCRB}$$

In this way, we do not solve the problem of saturability of the bound, but at least we can properly compare different strategies since the set of $\mathbb{R}$ is fully ordered.

Indeed, we can define the most informative bound V by an explicit minimization or a simple copy

$$C^{MI}[\underset{\rightarrow}{\theta}] = \min_{POVM} \left\{ \mathrm{tr}\left\{ \mathcal{F}(\underset{\rightarrow}{\theta})^{-1} \right\} \right\} \rightarrow$$ It is very hard to compute (no idea in general how to do it)

We thus have the following chain of inequality

$$\mathrm{tr}\{\mathrm{Cov}[\hat{\underset{\rightarrow}{\theta}}]\} \geq C^{MI}[\underset{\rightarrow}{\theta}] \geq C^{SLD}[\underset{\rightarrow}{\theta}]$$

C^{MI}: Hard to compute and by construction is attainable

C^{SLD}: $\hookrightarrow$ easy to compute but in general non attainable even in principle.

So here comes to rescue the Holevo Cramer Rao bound $C^{H}[\underset{\rightarrow}{\theta}]$.

This is a tighter bound than SLD-QCR i.e.

$$\mathrm{tr}\{\mathrm{Cov}[\hat{\underset{\rightarrow}{\theta}}]\} \geq C^{MI}[\underset{\rightarrow}{\theta}] \geq C^{H}[\underset{\rightarrow}{\theta}] \geq C^{SLD}[\underset{\rightarrow}{\theta}]$$

$C^{H}[\underset{\rightarrow}{\theta}]$ is defined by the following minimization

$$C^{H}[\underset{\rightarrow}{\theta}] = \min_{\substack{V \in S_d \\ \hat{\underset{\rightarrow}{X}} \in \mathcal{X}_{\underset{\rightarrow}{\theta}}}} \left\{ \mathrm{tr}\{V\} \mid \underset{\substack{\downarrow \\ \text{matrix inequality}}}{V \geq Z[\hat{\underset{\rightarrow}{X}}]} \right\}$$

where
- S_d set of real-symmetric $d \times d$ matrices
- $Z[\hat{\underset{\rightarrow}{X}}]$ is an $d \times d$ matrix whose elements are

$$Z[\hat{\underset{\rightarrow}{X}}]_{ij} = \mathrm{tr}\{X_i X_j \rho_{\underset{\rightarrow}{\theta}}\}$$

- $\mathcal{X}_{\underset{\rightarrow}{\theta}} = \{ \hat{\underset{\rightarrow}{X}} = (\hat{X}_1, \dots, \hat{X}_d) \mid X_i$ is hermitian, $\mathrm{tr}\{X_i \partial_j \rho_{\underset{\rightarrow}{\theta}}\} = \delta_{ij} \}$

This is regarded as the most informative bound since it has been proved that it can be attained in the asymptotic limit, i.e. when we perform collective measurement on an asymptotically large number of copies, i.e. $\rho_{\vec{\theta}}^{\otimes N}$ with $N \to +\infty$. This can be proven using the theory of Quantum local asymptotic normality.

For this reason in general we have $C^{MI}[\vec{\theta}] > C^{H}[\vec{\theta}]$

MI $\longrightarrow$ $\rho_{\vec{\theta}}$ $\longrightarrow$ single-shot

HCR $\longrightarrow$ $N \begin{cases} \rho_{\vec{\theta}} \\ \vdots \\ \rho_{\vec{\theta}} \end{cases}$ $\longrightarrow$ collective measurement in the asymptotic limit $N \to +\infty$

Some properties of the HCR bound:

- Hard to compute analitically, only few general examples:

  1) Single Qubit
  2) Gaussian states $\to$ it can be proven that they can be saturated on single shot with Gaussian measurement

     $\hookrightarrow$ this is important for QLAN

- Semidefinite program to compute it $\to$ standard computational technique

- Need collective measurement to saturate it

- For pure state is attainable on single shot measurement

So far we did not tackle the question of the saturability of the SLD - QCR bound. However, now that we know what is the most fundamental bound, i.e. the HCR bound $C^H[\underset{\rightarrow}{\theta}]$ we can ask what are the conditions such that

$$C^H[\underset{\rightarrow}{\theta}] \overset{?}{=} C^{SLD}[\underset{\rightarrow}{\theta}]$$

It turns out (Proof long and boring) that the conditions of saturability is known as the <u>weak</u> <u>compatibility</u> <u>condition</u>

$$\forall_{i,j} \quad \mathrm{tr}\{\rho_{\underset{\rightarrow}{\theta}} \ [\hat{L}_i, \hat{L}_j]\} = 0 \qquad 1$$

We see thus that the SLD - QCR can be saturate in the asymptotic limit of collective measurement if the SLDs commute on <u>average</u>.

This is a weaker condition than the full commutation of the SLDs, i.e.

$$2 \qquad [\hat{L}_i, \hat{L}_j] = 0$$

maybe necessary → open question

Indeed this condition is sufficient to saturate the MI, i.e. the bound on single copy of the state $\rho_{\underset{\rightarrow}{\theta}}$.

So we have the three following scenarios

- $C^{MI} > C^H > C^{SLD}$ → SLD QCR not attainable
- weak compatibility cond $\overset{1}{\Rightarrow}$ $C^{MI} > C^H = C^{SLD}$ →
  → SLD QCR attainable in the asymptotic limit of collective measurements
- compatibility ^{2} ⇒ $C^{MI} = C^H = C^{SLD}$ → SLD QCR attainable single copy

In the worst scenario ie

$$C^{MI} > C^{H} > C^{SLD}$$

one can actually prove that $C^{SLD} \leq C^{H} \leq (1+R)\, C^{SLD}$

where $0 \leq R \leq 1$ $\Rightarrow$ this imply that any relevant scaling in the HCR bound can be inferred from the C^{SLD} bound $\Rightarrow$

$\Rightarrow$ evaluation SLD-QFIM is still useful!

③ Quantum Local Asymptotic Normality

In this part of the lecture, I'd like to give some intuition on why the HCR bound is attainable in the asymptotic limit with collective measurements.

To do so:

- Single parameter $\rightarrow$ $|\psi_{\vec{\theta}}\rangle = e^{i\theta G}|\psi_0\rangle$ $\quad \langle\psi_0|G|\psi_0\rangle = 0$

$F_Q = 4\langle\psi_0|G^2|\psi_0\rangle$

- We assume that $\theta = \theta_0 + \frac{u}{\sqrt{n}}$ $\Rightarrow$ local parameter estimation
  u is unknown and random

$\Downarrow$

we are saying: we are close to θ_0 and the uncertainty is of order of the statistical uncertainty i.e. $\propto \frac{1}{\sqrt{n}}$

- Joint state $|\psi_u^n\rangle = |\psi^{\otimes n}_{\theta_0 + \frac{u}{\sqrt{n}}}\rangle$
- QFI of $|\psi_u^n\rangle$ for parameter "u" is F_Q $\rightarrow$ exercise, use change of parametrisation rule
- A pure state model $\longleftrightarrow$ family of Hilbert space vectors $\Longrightarrow$

$\Rightarrow$ a pure state model is uniquely determined by inner products of pairs of vectors with different parameters

$\Downarrow$

⊛ WEAK CONVERGENCE of STATISTICAL MODEL

A sequence of models converge to a limit model if overlaps converge pointwise

Now, we can easily see that $\rightarrow$

$$\langle \psi_u^n | \psi_v^n \rangle = \langle \psi_0 | \exp\left\{ \frac{i(u-v)G}{\sqrt{n}} \right\} | \psi_0 \rangle^n =$$

$$\| $$

$$1 + i\frac{(u-v)}{\sqrt{n}} G - \frac{(u-v)^2}{2n} G^2 + O(n^{-3/2})$$

$4 \langle \psi_0 | G^2 | \psi_0 \rangle = F_Q$

$$\overset{)}{=} \left(1 - \frac{(u-v)^2}{2n} \frac{F_Q}{4} + O(n^{-3/2}) \right)^n \xrightarrow[n \to +\infty]{}$$

$$\xrightarrow[n\to+\infty]{} \exp\left\{ - \frac{(u-v)^2 F_Q}{8} \right\}$$

Ok, why is this important?

If we introduce the quantum Gaussian shift model, i.e. coherent states of the form

$$\left| \sqrt{\frac{F_Q}{2}}\, u \right\rangle = e^{-iu\sqrt{\frac{F_Q}{2}} P} |0\rangle$$

↳ vacuum state

parameter

canonical coordinate

Exercise → • show that $\langle Q \rangle = \sqrt{\frac{F_Q}{2}}\, u$, $\langle P \rangle = 0$

↳ this is why we call shift model

• QFI for $u \rightarrow F_Q$

Furthermore → $\left\langle \sqrt{\frac{F_Q}{2}}\, u \middle| \sqrt{\frac{F_Q}{2}}\, v \right\rangle = \exp\left\{ - \frac{(u-v)^2}{8} F_Q \right\}$!

This means that we have proved that

$$\langle \psi^n_u | \psi^n_v \rangle \xrightarrow[n\to+\infty]{} \left\langle \sqrt{\frac{F_Q}{2}}\, u \middle| \sqrt{\frac{F_Q}{2}}\, v \right\rangle$$ $\rightsquigarrow$ they have the same QFI!

weakly convergence of stat. model

Equivalence of statistical model, why it is important?

At the single-parameter level $\rightarrow$ not too much, we know QCR bound is sotinable.

But this idea keep working also for pure $\forall$ multiparameter statistical model (and to some extent to mixed states, there are some caveats, one need to introduce a stronger version of convergence)

$\downarrow$

i.e. $$|\psi_{\underset{\rightarrow}{\theta}}\rangle = e^{\frac{i}{2}(\theta_1\sigma_x - \theta_2\sigma_y)}|0\rangle \qquad \underset{\rightarrow}{\theta} = (\theta_1, \theta_2) \Rightarrow \underset{\rightarrow}{\theta} = \underset{\rightarrow}{\theta}^0 + \frac{\vec{u}}{\sqrt{n}}$$

$$|\psi^n_{\underset{\rightarrow}{u}}\rangle = |\psi^{\otimes n}_{\vec{u}/\sqrt{n}}\rangle = \left(e^{\frac{i}{2}(u_2\sigma_x - u_1\sigma_y)/\sqrt{n}}|0\rangle \right)^{\otimes n}$$

The SLD for u_1 and u_2 is simply given as

$$L_1 = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sigma_x^{(i)} \quad \text{and} \quad L_2 = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sigma_y^{(2)}$$ $\longrightarrow$ exercise!

- They do not commute, even on average!

$\downarrow$

We can prove that

$$\langle \Psi^n_{\vec{u}} | \Psi^n_{\vec{v}} \rangle \xrightarrow{n\to\infty} \left\langle \frac{1}{\sqrt{2}} \vec{u} \middle| \frac{1}{\sqrt{2}} \vec{v} \right\rangle$$ → coherent states

↳ $|\Psi_{\vec{u}}\rangle = e^{\frac{i}{\sqrt{2}}(u_2 Q - u_1 P)} |\vec{0}\rangle$

Furthermore one can show that $L_1^{(n)}$ and $L_2^{(n)}$ converge to the SLDs of the Gaussian model, i.e. $\sqrt{2}\, Q$ and $\sqrt{2}\, P$ at $\vec{u} = 0$

This means that evaluating the optimal estimation of u_1, u_2 for $|\frac{1}{\sqrt{2}} \vec{v}\rangle$

⇒ corresponds to optimal estimation of $u_1 u_2$ for statistical model $|\Psi^n_{\vec{u}}\rangle$ made of qubits! The two statistical model are equivalent

One can show that HCR for $|\Psi_{\vec{u}}\rangle$ is equal to HCR bound for $|\frac{1}{\sqrt{2}} \vec{u}\rangle$ (HCR for Gaussian state can be analytically evaluated)

⇒ HCR for $|\Psi_{\vec{u}}\rangle$ is attainable using collective measurement on $|\Psi^n_{\vec{u}}\rangle$

We can summarise

1) $|\Psi^n_{\vec{u}}\rangle \longrightarrow$ converge to Gaussian shift model

2) HCR for Gaussian shift model can be saturated on single copy with Gaussian measurement.

3) HCR for $|\Psi_{\vec{u}}\rangle$ = HCR for $|\frac{1}{\sqrt{2}} \vec{u}\rangle$

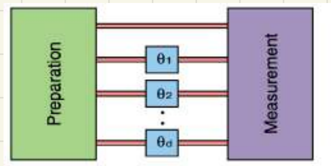
⇓

HCR for $|\Psi_{\vec{u}}\rangle$ can be obtained by collective measurements on $|\Psi^n_{\vec{u}}\rangle$ with $n \to +\infty$.

④ Quantum Enhanced Multiparameter Metrology

So far we have explored quantum-enhanced sensing with NOON states and entangled qubit states. Further, we have seen the fundamental bounds for multiparameter quantum metrology, namely HCR and SLD-QCR.

We have not explored if there is any quantum enhancement specific to multiparameter quantum metrology. What I would like to do now is exactly exploring this question: do we have any quantum practical advantage in estimating simultaneously multiple parameters?



Our setting is the following: we have $d+1$-mode interferometer (need $d+1$ to have a reference, otherwise singular model: only phase difference is measurable)

Our resource is the number of photons $\rightarrow N$: what's the best way to do that?

The most general state is $|\psi\rangle = \sum_{k=1}^{D} \alpha_k |N_{k,0},\dots, N_{k,d+1}\rangle = \sum_k \alpha_k |\vec{N}_k\rangle$

s.t. $\sum_{m=0}^{d} N_{k,m} = N \quad \forall k$, $\quad D = \dfrac{(N+d)!}{N!\,d!}$ $\rightarrow$ $\left\{ \text{number of distinct configurations of distributing } N \text{ photons across } d+1 \text{ modes} \right.$

$\downarrow$ for each element of the superposition

Unitary that encodes the parameters $\rightarrow U_\theta = \exp\{i \sum_{m=1}^{d} \hat{N}_m \theta_m\}$

The final state $\rightarrow |\psi_{\vec\theta}\rangle = \sum_{k=1}^{D} \alpha_k \, e^{i \vec{N}_k \cdot \vec{\theta}} |\vec{N}_k\rangle$

- Exercise $\rightarrow$ $I_{\vec\theta} = 4 \sum_i |\alpha_i|^2 \vec{N}_i \vec{N}_i^T - 4 \sum_{i,j} |\alpha_i|^2 |\alpha_j|^2 \vec{N}_i \cdot \vec{N}_j^T$

- Exercise $\rightarrow$ Evaluate SLD and show that weak compati bility is satisfied

I.e. $\langle \psi_{\underset{\to}{\theta}} | [L_m, L_n] | \psi_{\underset{\to}{\theta}} \rangle = 0$

This implies that HCR = SLD - QCR $\Rightarrow$ we do not have to evaluate HCR!

We have already seen (Saulo) that the optimal state to infer a single parameter θ is the NOON state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|N,0\rangle + |0,N\rangle \right) \rightsquigarrow \mathrm{Var}[\tilde{\theta}] = \Delta\theta^2 \sim \frac{1}{N^2}$$ ~ Heisenberg scaling!

Here we have $d+1$ modes $\rightsquigarrow$ we can extend NOON state to multi mode state

$$\rightsquigarrow |\psi\rangle = \alpha \left(|0,N,\dots,0\rangle + \dots + |0,0,\dots,N\rangle \right) + \beta\, |N,0,\dots,0\rangle$$

$$\Rightarrow [I_{\underset{\to}{\theta}}]_{\ell,m} = 4N^2 \left(\delta_{\ell,m}\, \alpha^2 - \alpha^4 \right) \Rightarrow$$

$$\min_{\alpha} \mathrm{tr}\{ I_{\underset{\to}{\theta}}^{-1} \} = \frac{(1+\sqrt{d})^2}{N^2} \frac{d}{4} \qquad \alpha = \frac{1}{\sqrt{d+\sqrt{d}}}$$

$$|\Delta \underset{\to}{\theta}_S|^2 = \mathrm{tr}\{\mathrm{Cov}[\underset{\to}{\tilde{\theta}}]\} = \sum_{i=1}^{d} \Delta\theta_i^2 \sim \frac{1}{\nu} \frac{(1+\sqrt{d})^2}{N^2} \frac{d}{4}$$ with ν number of samples

Is there any advantage with individual estimation? To understand this, let compare this bound with the separate estimation of d parameters

Optimal equivalent individual strategy is obtained with NOON state $\Rightarrow$

Given that our resource is N photons $\Rightarrow$ we allocate N/d photon per parameter (i.e. per experiment)

Recalling that $\Delta\theta^2 \sim \frac{1}{\nu}\frac{1}{N}$ $\Rightarrow$ $\forall i=1,\dots,d$ $\Delta\theta_i^2 \sim \frac{1}{\nu}\frac{d^2}{N^2}$

So the scalar bound reads as $|\Delta\vec{\theta}_{ind}^2| = \sum_{i=1}^{d} \Delta\theta_i^2 \sim \frac{1}{\nu}\frac{d^3}{N^2}$

We can also compare the situation with uncorrelated coherent state

$\bigotimes_{i=1}^{d} |\alpha_i\rangle$ with $\sum_{i=1}^{d} \langle\alpha_i|\,\hat{N}_i\,|\alpha_i\rangle = N$ $\Rightarrow$ $\Delta\theta_i^2 \sim \frac{1}{\nu}\frac{d}{N}$ $\Rightarrow$

$\Rightarrow$ $|\Delta\vec{\theta}_d^2| = \sum_{i=1}^{d} \Delta\theta_i^2 = \frac{d^2}{\nu N}$

So we have $\rightarrow$

- simultaneous estimation $\sim \frac{(1+\sqrt{d})^2}{N^2}\frac{d}{4}$
- individual estimation $\sim \frac{d^3}{N^2}$
- classical estimation $\sim \frac{d^2}{N}$

$$\frac{|\Delta\vec{\theta}_s|^2}{|\Delta\vec{\theta}_{ind}|^2} = \frac{1}{\cancel{N^2}}(1+\sqrt{d})^2\frac{\cancel{d}}{4}\cdot\frac{\cancel{N^2}}{d^{\cancel{3}2}} = \frac{(1+\sqrt{d})^2}{4d^2} = \frac{d}{4d^2}\left(1+\frac{1}{\sqrt{d}}\right) \sim$$

$\Rightarrow$

$$\sim \frac{1}{4d} \Rightarrow O(d) \text{ advantage!}$$

A simultaneous strategy has an $O(d)$ advantage wrt to individual strategy

⊛ Quantum advantage for multiparameter quantum metrology.

- Since pure state attainable on single copy level!
  Quantum advantage not only in the resource N but also in the number of parameter $\rightarrow$ cost $\rightarrow$ more complicated protocol!

Caveats $\rightarrow$ state hard to implement $\rightarrow$ other states with suboptimal advantage wrt to multimode NOON but still better than individual estimation with NOON

Conclusion

Quantum can enhance but also limits precision in parameter estimation

Quantum mechanics can limit simultaneous estimation on multiple parameter : this is a consequence of incompatibility of optimal measurement
↳ "quantum noise". ⇒ Structure of quantum multiparameter bound is much more complicated (there are other bounds I did not mention). The the question of saturability of these bound is also more complicated to address.

On the other hand, we have seen that quantum enhanced metrology at single and multiparameter metrology is possible : application to imaging, biological sensing, gravitational waves detector and much more !