Introduction to quantum thermometry

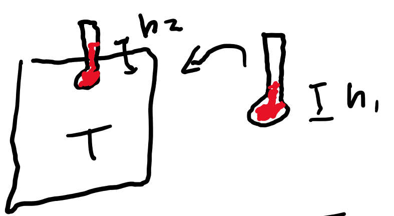
Why quantum thermometry? (J. PHYS. A 52, 303001 (2019))

- Quantum technologies ($T \sim mK$)
- Ultracold atomic experiments ($T \sim nK$)

measuring very low temperatures is hard

Figuring out fundamental limits important

How do we measure temperature?



If I know $h(T)$

$\rightarrow$ can find T from h

We find T by measuring something we know how depends on T.

What is temperature in quantum?

$$\hat{\rho}_T = \frac{1}{Z} e^{-\hat{H}/T} = \frac{1}{Z} \sum_k e^{-E_k/T} |E_k\rangle\langle E_k|$$

$$\hat{H} = \sum_k E_k |E_k\rangle\langle E_k| , \quad Z = tr\left(e^{-\hat{H}/T}\right), \quad k_B = 1$$

What are the fundamental limits to precision?

Want to measure T from $\hat{\rho}_T$ and a set of POVMs $\{\hat{\pi}_x\}$ $\left(\sum_x \hat{\pi}_x = 1\right)$

$$\hat{\pi}_x \rightarrow \{x_1, x_2, \ldots, x_n\} = \vec{x}$$

$$p(x) = Tr\left(\hat{\pi}_x \hat{\rho}_T\right)$$

Construct an estimator:

(Max likelihood)

construct an estimator:

$$E\left[T_{est}(\vec{x})\right] = T \quad \text{(Max. likelihood)}$$

↑ unbiased

$$\Delta T^2 = E\left[\left(T_{est}(\vec{x}) - T\right)^2\right] \geq \frac{1}{M F(\hat{\Pi}_x, \hat{\rho}_T)}$$

$$F(\hat{\Pi}_x, \hat{\rho}_T) = \sum_x P(x)\left(\frac{\partial \ln(P(x))}{\partial T}\right)^2 \leftarrow \text{Fisher info (FI)}$$

$$\mathcal{F}(\hat{\rho}_T) = \max_{\{\hat{\Pi}_x\}} F(\hat{\Pi}_x, \hat{\rho}_T) \leftarrow \text{Quantum FI (QFI)}$$

we want to measure T from $\hat{O}$

$$\Delta T^2 = \frac{\Delta\hat{O}^2}{|\partial_T\langle\hat{O}\rangle|^2}, \quad \Delta\hat{O}^2 = \langle\hat{O}^2\rangle - \langle\hat{O}\rangle^2$$

$$\partial_T\langle\hat{O}\rangle = \frac{\partial\langle\hat{O}\rangle}{\partial T}$$

$$\langle\hat{O}\rangle = \mathrm{Tr}(\hat{O}\hat{\rho}_T)$$

Example:

$$\hat{H} = \varepsilon\hat{\sigma}_z = \varepsilon(|0\rangle\langle 0| - |1\rangle\langle 1|)$$

$$\hat{\rho}_T = \frac{1}{Z}\left(e^{-\varepsilon/T}|0\rangle\langle 0| + e^{\varepsilon/T}|1\rangle\langle 1|\right)$$

$$Z = 2\cosh(\varepsilon/T)$$

want to find best $\hat{O}$ to get T.

Maximize $\partial_T\langle\hat{O}\rangle$

$$\partial_T\langle\hat{O}\rangle = \mathrm{Tr}(\hat{O}\,\partial_T\hat{\rho}_T)$$

$$\partial_T\hat{\rho}_T = \frac{\varepsilon}{2T^2\cosh^2(\varepsilon/T)}(|0\rangle\langle 0| - |1\rangle\langle 1|)$$

$$= \frac{1}{T^2}(\hat{H} - \langle\hat{H}\rangle)\hat{\rho}_T$$

(Check this

$= \frac{1}{T^2} (\ldots$

Show this

$$\partial_T \langle \hat{O} \rangle = \frac{1}{T^2}\left[\mathrm{Tr}(\hat{O}\hat{H}\hat{\rho}_T) - \langle \hat{H} \rangle \mathrm{Tr}(\hat{O}\hat{\rho}_T) \right]$$
$$= \frac{1}{T^2}\left[\langle \hat{O}\hat{H} \rangle - \langle \hat{O} \rangle \langle \hat{H} \rangle \right]$$
$$= \frac{1}{T^2} \operatorname{cov}(\hat{O}, \hat{H})$$

Best $\hat{O} = \hat{H}$ (!)

This is general

Show this for $\hat{H} = \sum_n E_n |E_n\rangle\langle E_n|$

How precisly can I infer T?

$$M \Delta T = \frac{\Delta \hat{H}^2}{|\partial_T \langle \hat{H} \rangle|^2} = \frac{\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2}{(\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2)^2} T^4$$
$$= \frac{T^4}{\Delta \hat{H}^2} \qquad \left(\mathcal{F} = \frac{\Delta \hat{H}^2}{T^4} \right)$$

This is still general.

Qubit:

$$\langle \hat{H} \rangle = \frac{\varepsilon}{Z}\left(e^{-\varepsilon/T} - e^{\varepsilon/T} \right)$$
$$\langle \hat{H}^2 \rangle = \frac{\varepsilon^2}{Z}\left(e^{-\varepsilon/T} + e^{\varepsilon/T} \right)$$
$$\Rightarrow M \Delta T^2 = \frac{T^4 \cosh^2(\varepsilon/T)}{\varepsilon^2}$$

Relative precision:

$$M \frac{\Delta T^2}{T^2} = \frac{T^2}{\varepsilon^2} \cosh^2(\varepsilon/T)$$

What happens for small T?

$$\ldots \; e^{-\varepsilon/T} + e^{\varepsilon/T} \sim e^{\varepsilon/T}$$

What happens for s

$\frac{T}{\varepsilon} << 1$, $\cosh(\varepsilon/T) \sim e^{-\varepsilon/T} + e^{\varepsilon/T} \sim e^{\varepsilon/T}$

Exponential divergence!

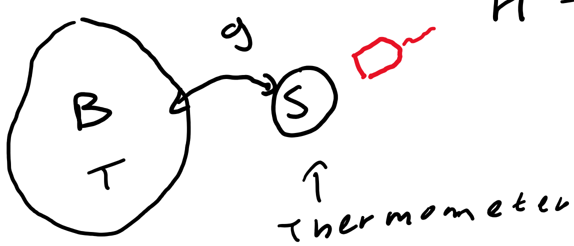
This is true for any (-ish) $\hat{H}$

Other Problems:

- After measurement system destroyed ($|E_k\rangle\langle E_k|$)
- Getting the full spectrum of $\hat{H}$ is hard.

<u>Solution to two of these</u>

$$\hat{H} = \hat{H}_B + \hat{H}_S + g H_{SB}$$



S and B in equilibrium

$$\hat{\rho}_{SB} = \frac{1}{Z} e^{-\hat{H}/T}$$

$$\hat{\rho}_S = Tr_B(\hat{\rho}_{SB}) \quad \text{(hard)}$$

g small, B is large

$$\hat{\rho}_{SB} = \hat{\rho}_S \otimes \hat{\rho}_B, \quad \hat{\rho}_S = \frac{1}{Z_S} e^{-\hat{H}_S/T}$$

$$\hat{\rho}_B = \frac{1}{Z_B} e^{-\hat{H}_B/T}$$

Example:

S is a qubit

We already solved this!

By measuring S we can get T without destroying B ✓

Getting the spectrum of S is easy ✓

Divergence for low T ✗

N-level Probe

$$\hat{H} = \sum_{n=1}^{N} E_n |E_n\rangle\langle E_n|$$

$$\mathcal{F} = \frac{\Delta \hat{H}^2}{T^4}$$

can we find $\{E_n\}$ to maximize $\Delta \hat{H}^2 \to \mathcal{F}$ (QFI)

$$\frac{\partial \Delta \hat{H}^2}{\partial E_n} = 0 \qquad n = 1, 2, \ldots, N$$

$$\langle \hat{H} \rangle = \sum_n E_n e^{-E_n/T}$$

$$\langle \hat{H}^2 \rangle = \sum_n E_n^2 e^{-E_n/T}$$

$$\frac{\partial \langle \hat{H}^2 \rangle}{\partial E_n} = \left(2E_n - \frac{E_n^2}{T}\right) e^{-E_n/T}$$

$$\partial \langle \hat{H} \rangle^2 \quad \ldots \quad \left(1 - \frac{E_n}{T}\right) e^{-E_n/T}$$

∂E_n

$$\frac{\partial \langle \hat{H} \rangle^2}{\partial E_n} = 2\langle \hat{H} \rangle \left(1 - \frac{E_n}{T}\right) e^{-E_n/T}$$

$\Rightarrow$ N coupled eqs.

$$E_n\left(2 - \frac{E_n}{T}\right) - 2\langle \hat{H} \rangle \left(1 - \frac{E_n}{T}\right) = 0$$

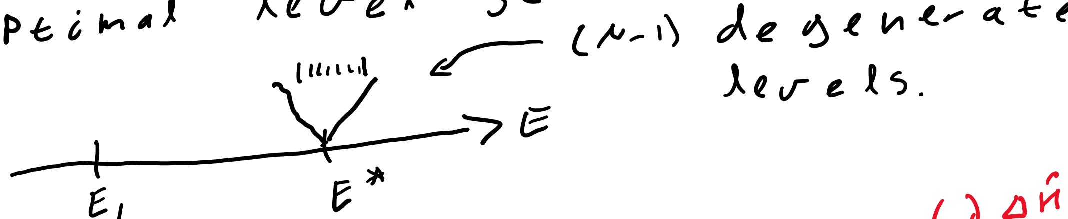
subtract eq. for m from eq. for n

$$\Rightarrow (E_n - E_m)\left(2 + \frac{2\langle \hat{H} \rangle}{T} - \frac{E_n + E_m}{T}\right) = 0$$

$$\Rightarrow E_n = E_m \quad \text{or} \quad E_n + E_m = 2T + 2\langle \hat{H} \rangle$$

$\Rightarrow$ The optimal probe is a two-level system with degenerate levels

Optimal level structure



Show that this is maximum $\left(\frac{\partial \Delta \hat{H}^2}{\partial E_n \partial E_m}\right)$

Show that $e^x = (N-1)\frac{x+2}{x-2}$ $\quad x = \frac{E^*}{T}$

Did we really do better?

$$\mathcal{F} = \frac{\Delta \hat{H}^2}{T^4} \qquad (\text{set } E_1 = 0)$$

$$\langle \hat{H} \rangle = (N-1) E^* e^{-x}$$

$$\langle \hat{H}^2 \rangle = (N-1)(E^*)^2 e^{-x}$$

to $\mathcal{F}$ use e^x from above

$\langle \hat{H}^2 \rangle = (N-1)(E \quad) -$

Insert this to $\mathcal{F}$, use e^x from above
we can show:

$$\mathcal{F} = \frac{x^2 e^x}{T^2} \frac{N-1}{(N-1+e^x)^2}$$

Not very intuitive...

Is ΔT^2 still diverging?

Look at the limit $N \to \infty$

$$e^x = (N-1)\frac{x+2}{x-2} \Rightarrow x \gtrsim \ln N$$

$$M\frac{\Delta T^2}{T^2} = \frac{(N-1+e^x)^2}{(N-1)x^2 e^x} \sim \frac{4N^2}{N^2 x^2} \sim \frac{4}{(\ln N)^2}$$

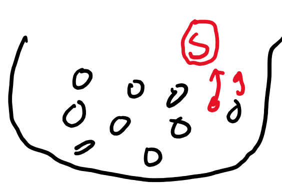
No divergence! (We cheated...)

order of limits. $x = E^*/T$

$$T \to 0: \quad \frac{(N-1+e^x)^2}{(N-1)x^2 e^x} \sim e^{E^*/T}$$

<u>But</u> for fixed T, we can scale error as $\sim 1/(\ln N)^2$

Non-equilibrium probes



$\sim \hat{\rho}_T$

Want to find T by looking at $\hat{\rho}_S(t)$.

Want to find T by looking at $\rho_s(t)$.

Spin-boson model

$$\hat{H} = \hat{H}_B + \hat{H}_S + \hat{H}_{SB}, \quad \hat{H}_B = \sum_k \omega_k b_k^\dagger b_k,$$

$$\hat{H}_S = \frac{\omega_0}{2}\hat{\sigma}_z, \quad \hat{H}_{SB} = \hat{\sigma}_z \sum_k (g_k b_k^\dagger + g_k^* b_k)$$

Initially: $\hat{\rho}_{SB}(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_T$

$$\hat{\rho}_S(0) = |+\rangle\langle+| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

One can show in interaction picture:

$$\hat{\rho}_S(t) = \mathrm{Tr}_B\left(\hat{U}_I(t)\, \hat{\rho}_{SB}\, \hat{U}_I^\dagger\right) = \frac{1}{2}\begin{pmatrix} 1 & e^{-\Gamma(t)} \\ e^{-\Gamma(t)} & 1 \end{pmatrix}$$

($[\hat{H}_S, \hat{H}_{SB}] = 0$, energy of probe conserved)

Again, one can show:

$$\mathcal{F} = 2\sum_{n,m} \frac{|\langle r_n|\partial_T \rho_S|r_m\rangle|^2}{r_n + r_m}$$

$$\hat{\rho}_S|r_n\rangle = r_n|r_n\rangle$$

$$r_\pm = \frac{1}{2}\left(1 \pm e^{-\Gamma(t)}\right)$$

$$|r_\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)$$

$\vdots$ plug this into $\mathcal{F}$

$$\mathcal{F} = \frac{|\partial_T \Gamma(t)|^2}{e^{2\Gamma(t)} - 1},$$ $\mathcal{F}$ depends on time.

Let's say $\Gamma(t) = \alpha T t$ (high T)

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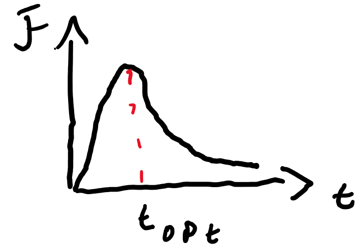
$$\Rightarrow \bar{J} = \frac{(\alpha t)^2}{e^{2\alpha T t} - 1}$$



- There is an optimal time for measurement that depends on T
- The optimal measurement basis (at $t = t_{opt}$) depends on T.

General features of non-equilibrium thermometry.

→ Local thermometry, requires previous knowledge of T.

Other directions of thermometry:

- Global thermometry
- Bayesian schemes
- Quantum resources (Non-Markovianity, correlations, etc.)
- Phase transitions
- Much more!