The Kraus representation of CPTP maps

<u>Thm</u> A map $\mathcal{E}: \mathcal{S} \to \mathcal{S}$ is CPTP if and only if there exists a set $\{M_x\}_x$ of operators such that

$$\mathcal{E}(\rho) = \sum_x M_x \rho M_x^\dagger \quad \text{and} \quad \sum_x M_x^\dagger M_x = \mathbb{I}$$

The action of a map may be represented by taking the expectation value on the stochastic application of the Kraus operators, with the appropriate probability. Indeed:

$$\mathcal{E}(\rho) = \sum_x M_x \rho M_x^\dagger = \sum_x \mathrm{Tr}[M_x \rho M_x^\dagger] \frac{M_x \rho M_x^\dagger}{\mathrm{Tr}[M_x \rho M_x^\dagger]} = \mathbb{E}_x \frac{M_x \rho M_x^\dagger}{\mathrm{Tr}[M_x \rho M_x^\dagger]}$$

because $\mathrm{Tr}[M_x \rho M_x^\dagger]$ is a valid probability distribution, as

$$p_x = \mathrm{Tr}[M_x \rho M_x^\dagger] = \mathrm{Tr}[M_x^\dagger M_x \rho] = \langle M_x^\dagger M_x, \rho \rangle_{HS} \geq 0 \text{ as they are both positive}$$

$$\sum_x p_x = \sum_x \mathrm{Tr}[M_x^\dagger M_x \rho] = \mathrm{Tr}\Big[\underbrace{\sum_x M_x^\dagger M_x}_{\mathbb{I}} \rho\Big] = \mathrm{Tr}[\rho] = 1 . \quad \square$$

If we apply the same map multiple times:

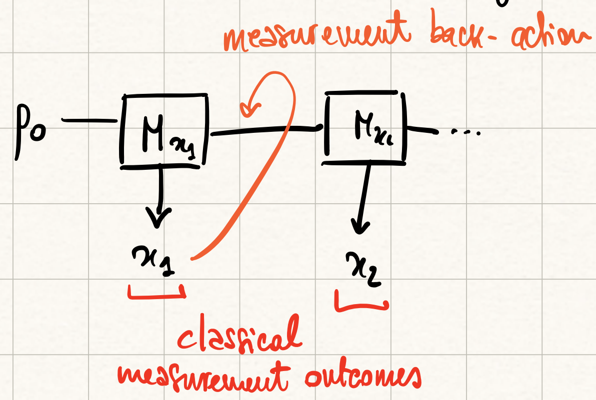
$$\mathcal{E}(\mathcal{E}(\rho)) = \sum_{x_1} \sum_{x_2} M_{x_2} M_{x_1} \rho M_{x_1}^\dagger M_{x_2}^\dagger =$$

$$= \sum_{x_1 x_2} \mathrm{Tr}[M_{x_2} M_{x_1} \rho M_{x_1}^\dagger M_{x_2}^\dagger] \frac{M_{x_2} M_{x_1} \rho M_{x_1}^\dagger M_{x_2}^\dagger}{\mathrm{Tr}[M_{x_2} M_{x_1} \rho M_{x_1}^\dagger M_{x_2}^\dagger]} =$$

$$= \mathbb{E}_{x_1 x_2} \frac{M_{x_2} M_{x_1} \rho M_{x_1}^\dagger M_{x_2}^\dagger}{\mathrm{Tr}[M_{x_2} M_{x_1} \rho M_{x_1}^\dagger M_{x_2}^\dagger]}$$

And so on. Moral of the story: we can reproduce the action of a map by stochastic application of its Kraus operators. A single sequence of outcomes is a quantum trajectory.

Now: note that each M_x operator can be seen as a POVM element : this formalism allows us to describe systems undergoing continuous measurements. That is:



Now: continuous time

Assume infinitesimal timesteps of duration $dt \ll 1$. Want to write a map that evolves ρ_t to ρ_{t+dt}, via its Kraus operators. With a single Kraus op. it would be a unitary, so assume two. We also want that nothing happens for $dt=0$, hence:

$$M_0 = \mathbb{1} - \frac{1}{2} L^\dagger L\, dt - iH dt \qquad M_1 = ?$$

$$M_0^\dagger M_0 + M_1^\dagger M_1 = \mathbb{1} \Rightarrow M_1^\dagger M_1 = \mathbb{1} - M_0^\dagger M_0 =$$

$$= \mathbb{1} - \left(\mathbb{1} - \frac{1}{2} L^\dagger L\, dt + iH dt\right)\left(\mathbb{1} - \frac{1}{2} L^\dagger L\, dt - iH dt\right) =$$

$$= \mathbb{1} - \left(\mathbb{1} - \frac{1}{2} L^\dagger L\, dt + \cancel{iHdt} - \frac{1}{2} L^\dagger L\, dt - \cancel{iHdt}\right) = L^\dagger L\, dt$$

$$\Rightarrow M_1 = L\sqrt{dt}$$

So, we have the CPTP (to order dt) map
$$M_0 = \mathbb{1} - \frac{1}{2} L^\dagger L\, dt - iH dt$$
$$M_1 = L\sqrt{dt}$$

Average over trajectories (by hand for now)

$$\rho_{t+dt} = M_0 \rho M_0^\dagger + M_1 \rho M_1^\dagger =$$

$$= \left(\mathbb{I} - \frac{1}{2} L^\dagger L \, dt - iH dt \right) \rho \left(\mathbb{I} - \frac{1}{2} L^\dagger L \, dt + iH dt \right) + L \rho L^\dagger dt =$$

$$= \rho - \frac{1}{2} \{ L^\dagger L, \rho \} dt - i [H, \rho] dt + L^\dagger L \, dt$$

That is the GKSL master equation.

In time, we can think that at each step there is a stochastic choice of either M_0 or M_1.

In particular, M_0 is applied with probability

$$p_0^t = \mathrm{Tr}[M_0 \rho_t M_0^\dagger]$$

and M_1

$$p_1^t = \mathrm{Tr}[M_1 \rho_t M_1^\dagger]$$

We used to use x to represent the outcome at each step. Now, let us call it dN_t to represent its infinitesimal nature. Of course, $dN_t \in \{0, 1\}$, hence $dN_t^2 = dN_t$.

Furthermore:

$$\mathbb{E}[dN_t] = 0 \, p_0^t + 1 \, p_1^t = p_1^t = \mathrm{Tr}[M_1 \rho_t M_1^\dagger] = dt \, \mathrm{Tr}[L \rho L^\dagger]$$

Hence, with a bit of mathematical clumsiness, we say $dN_t = \mathcal{O}(dt)$.

Let us write our evolution on a trajectory as

$$\rho_{t+dt}^c = \underbrace{(1 - dN_t)}_{\text{if } M_0 \text{ is selected}} \frac{M_0 \rho_t M_0^\dagger}{\mathrm{Tr}[M_0 \rho_t M_0^\dagger]} + \underbrace{dN_t}_{\text{if } M_1 \text{ is chosen}} \frac{M_1 \rho_t M_1^\dagger}{\mathrm{Tr}[M_1 \rho_t M_1^\dagger]}$$

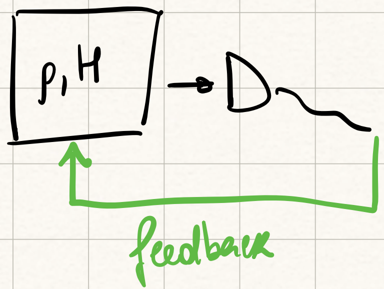
Stochastic master equation (SME)

$dN_t \rightarrow$ Poisson increment

Key property: $\mathbb{E}[\rho_{t+dt}^c]$ should yield the GKSL master equation.

Proof $\longrightarrow$ Albarelli, Genoni, Phys. Lett. A 2023 (e.g.)

Why should we care? It is a natural description of systems undergoing continuous monitoring via direct photodetection



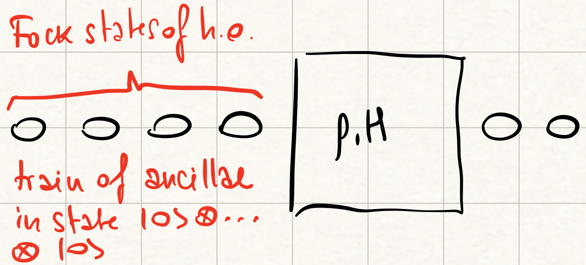
And then … we could close the loop, and act upon the system, based on the measurement (e.g. for the preparation of a state)

$\Rightarrow$ Hari's lecture 19/03

Led by this experimental insights, we can consider different measurement schemes, corresponding to different unravellings of the GKSL equation.

Collisional perspective

Alternatively, we can introduce trajectories by thinking of collisional models.



Interaction between system and environment given by the Hamiltonian H_{int}, given by

$$H_{int} = i \frac{\sqrt{\kappa}}{\sqrt{dt}} \left(a \otimes b_t^\dagger - a^\dagger \otimes b_t \right)$$

$\rightarrow$ rescaling of κ w. time

where a bosonic/fermionic ladder operator on the system, and b_t bosonic operator on the ancilla at time t.

$$\Rightarrow \quad U(t, t+dt) = \exp\left[\sqrt{\kappa dt} \left(a \otimes b_t^\dagger - a^\dagger \otimes b_t \right) \right]$$

(+ Hamiltonian of the system) $\Rightarrow \dots \Rightarrow$ yields the GKSL equation again.

But now, assume we measure the ancilla when it comes out.

- measure eigenstates $|n\rangle$ of $b_t^\dagger b_t$ $\Rightarrow$ photodetection, see above
- measure eigenstates $|x\rangle$ of $b_t + b_t^\dagger$ $\Rightarrow$ homodyne.
- ⋮ etc.

Let us consider the second case. Unnormalized conditional state of the system after interacting with ancilla at timestep t:

$$\tilde{\rho}^c_x(t+dt) = \mathrm{Tr}_t\Big[U(t, t+dt)\big(\rho(t) \otimes |0\rangle\langle 0|\big) U^\dagger(t, t+dt) \big(\mathbb{1} \otimes |x\rangle\langle x|\big)\Big] =$$

$$= \langle x| U(t,t+dt)\ \rho(t) \otimes |0\rangle\langle 0|\ U^\dagger(t,t+dt)|x\rangle$$

Probability: $p_x = \mathrm{Tr}\left[\tilde{\rho}^c_x(t+dt)\right]$

Expand U at order $\sqrt{dt}$:

$$U(t, t+dt) \sim \mathbb{1} \otimes \mathbb{1} + \sqrt{\kappa dt}\left(a \otimes b_t^\dagger - a^\dagger \otimes b_t\right) + o(\sqrt{dt})$$

$$\Rightarrow \tilde{\rho}^c_x(t+dt) = \langle x|\left(\mathbb{1} \otimes \mathbb{1} + \sqrt{\kappa dt}\left(a \otimes b_t^\dagger - a^\dagger \otimes b_t\right)\right) \rho(t) \otimes |0\rangle\langle 0| \,\cdot$$

$$\cdot \left(\mathbb{1} \otimes \mathbb{1} + \sqrt{\kappa dt}\left(a^\dagger \otimes b_t - a \otimes b_t^\dagger\right)\right)|x\rangle =$$

$$= |\langle x|0\rangle|^2 \rho(t) + \sqrt{\kappa dt}\left(a\rho(t)\langle x|1\rangle\langle 0|x\rangle + \rho(t)a^\dagger\langle x|0\rangle\langle 1|x\rangle\right) = (*)$$

Now, by using the shape of the eigenfunctions of the harmonic oscillator on the position basis (Hermite functions), define $p(x) = |\langle x|0\rangle|^2$, and $\langle x|1\rangle = \sqrt{2}\,x\sqrt{p(x)}$.

Then:

$$(*) = p_x(t)\left[\rho(t) + \sqrt{\kappa dt}\sqrt{2}\,x\left(a\rho(t) + \rho(t)a^\dagger\right)\right]$$

The probability is the trace of the unnormalised state: a Gaussian with variance $\frac{1}{2}$ and mean value $\sqrt{\frac{\kappa dt}{2}}\langle a+a^\dagger\rangle_{\rho(t)}$

⇒ deterministic component of mean + random Gaussian variable. Def.

$$dy_t = \sqrt{\kappa}\,\langle a+a^{\dagger}\rangle_t\,dt + dw_t \qquad \text{(infinitesimal change)}$$

with $Var(dw_t) = dt$, $\mathbb{E}[dw_t] = 0$

dw_t is the **Wiener increment**, and satisfies the rules of **Itô calculus**. The most important:

$$dw_t^2 = dt \qquad \text{(Itô rule)}$$

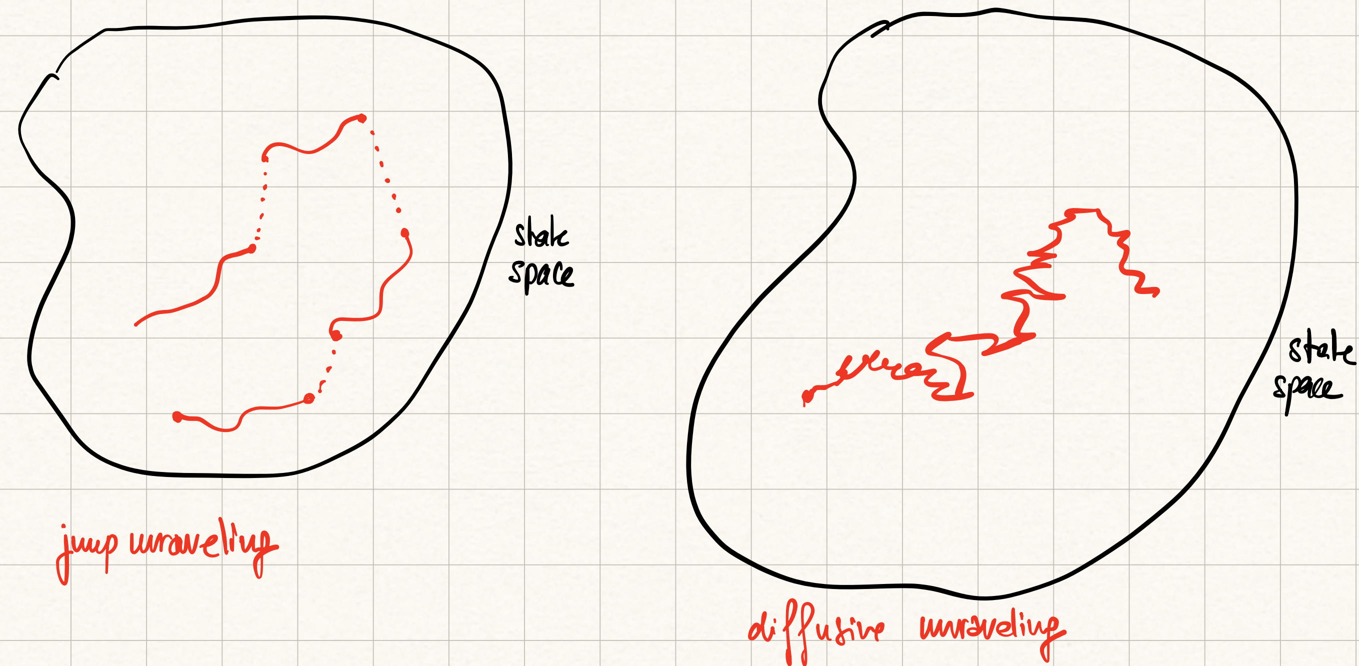
⇒ the infinitesimal current is the exp. value of the quadrature x, plus some white noise.

Obtain an equivalent SME: $\mathcal{H}[A]\,\bullet = A\bullet + \bullet A - \langle A + A^{\dagger}\rangle\bullet$

$$d\rho^{(c)}(t) = \sqrt{\kappa}\,\mathcal{H}[a]\rho(t)\,dw + o(\sqrt{dt})$$

and then obtain terms at order dt by imposing the GKSL eq. for average.

Recap



A more fundamental question: do systems exist? It depends on how they are observed!