Entanglement & Quantum Resource Theories

Motivation: Entanglement is useful!

Quantum teleportation

$|x\rangle|\phi^+\rangle_{AB}$ $\qquad\qquad$ $|x\rangle = c_0|0\rangle + c_1|1\rangle$

$\propto |00\rangle + |11\rangle$

Goal: Use operations in labs $S\otimes A$ & B to bring $|x\rangle$ to Bob's side.

1. Alice measures in Bell-basis $\{\phi^+, \phi^-, \psi^+, \psi^-\}$ on $S\otimes A$
2. Alice sends result r to Bob.
3. Bob corrects with U_r on B.

What is U_r for $r \in \{\phi^+, \phi^-, \psi^+, \psi^-\}$?

Superdense coding

$|\phi^+_{AB}\rangle$, classical message $m \in \{00, 01, 10, 11\}$

Goal: Share m with Bob by transforming only a single qubit.

Show that m can be recovered by Bob from $|\psi_m\rangle = U_m^{(A)} \otimes \mathbb{1}^{(B)} |\phi^+_{AB}\rangle$ by Bell-basis measurement after Alice transfers her part of the state.

$U_m \in \{\mathbb{1}, X, Y, Z\}$

Morale: Entanglement enables tasks not possible without it. It gets used up in the process.

<u>Entanglement is a resource.</u>

Quantum Resource Theories (QRTs)

Definition
A QRT is a tuple $R = (\mathcal{O}, \mathcal{F})$ where:
- $\mathcal{O}$: free operations
- $\mathcal{F}$: free states

Questions, e.g.:

- Given some ρ, what states r can it be <u>transformed</u> to under $\mathcal{O}$
  ↳ <u>assymptotic</u> conversion?
  → <u>Rate</u>: $r > \frac{n}{m}$

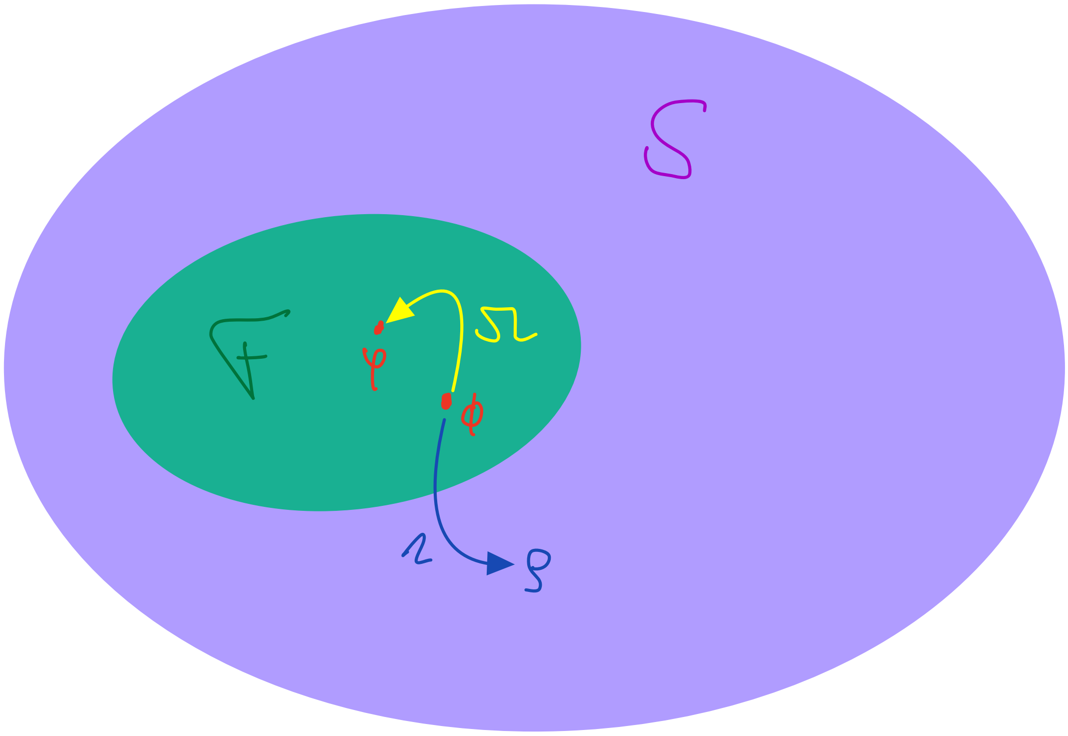
$$\rho^{\otimes m} \xrightarrow{\mathcal{O}} r^{\otimes n} ?$$

- <u>How much</u> resource is contained in $\rho \notin \mathcal{F}$?
- What <u>tasks</u> does $\rho \notin \mathcal{F}$ enable?

Free operations

Free operations postulate (FOP)

$$\forall \varphi = \Omega(\phi),\ \Omega \in \mathcal{O},\ \phi \in \mathcal{F} : \varphi \in \mathcal{F}$$



Recap: quantum instruments

$I = \{\Lambda_j\}$ Λ_j : CP $p(j|\rho) = tr[\Lambda_j(\rho)]$

trace preservation:

$$tr\left(\sum_j \Lambda_j(\rho)\right) = tr(\rho)$$

$$\rho_j = \frac{\Lambda_j(\rho)}{tr(\Lambda_j(\rho))}$$

$$I(\rho) = \sum_j \Lambda_j(\rho) \otimes |j\rangle\langle j|$$

Def.: (one-way LOCC $[\mathcal{O}_{1\text{-LOCC}}]$)

LOCC ↑ local operations and classical communication

Instrument $\mathcal{I} = \{F_j\}_j \in \mathcal{O}_{1\text{-LOCC}}$

$$\Leftrightarrow \exists\, \mathcal{I}^{(k)} = \{\mathcal{E}_j^{(k)}\}_j \text{ s.t.}$$

$$F_j = \bigotimes_{m<k} \Lambda_j^{(m)} \otimes \mathcal{E}_j^{(k)} \bigotimes_{n>k} \Lambda_j^{(n)}$$

with $\Lambda_j^{(e)}$ CPTP

Def. (LOCC $[\mathcal{O}_{LOCC}]$)

$\Lambda \in \mathcal{O}_{LOCC} \Leftrightarrow \Lambda$ can be implemented by a succession of 1-LOCC operations, where coarse-graining and conditioning on previous measurements is allowed.

Def. (separable states $[\mathcal{F}_s]$)

For $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \dots \otimes \mathcal{H}^{(N)}$:

$$\sigma = \sum p_i\, \sigma_i^{(1)} \otimes \sigma_i^{(2)} \dots \otimes \sigma_i^{(N)} \qquad p_i \geq 0,\ \sum p_i = 1$$

create σ by LOCC:

1) sample $\{p_i\}$ locally
2) broadcast i
3) prepare $\sigma_i^{(k)}$ at each site k.

Def. (seperable operations $[\mathcal{O}_{SEP}]$)

For $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \dots \otimes \mathcal{H}^{(N)}$:

CPTP map $\Lambda \in \mathcal{O}_{SEP}$

$\Leftrightarrow \exists$ Kraus representation with $K_i = \bigotimes_k^N A_i^{(k)}$

E.g. bipartite $\Lambda(\varrho) = \sum_i A_i \otimes B_i \, \varrho_{AB} A_i^\dagger \otimes B_i^\dagger$

Note: 1) $\mathcal{O}_{LOCC} \subset \mathcal{O}_{SEP}$

2) $\mathcal{F}_S$ closed under $\mathcal{O}_{SEP}$

$\forall \Lambda \in \mathcal{O}_{SEP}, \ \phi \in \mathcal{F}_S : \Lambda(\phi) \in \mathcal{F}_S$

Maximal set of free operations $\mathcal{O}_{max}$:

Def. (non-entangling operations [$\mathcal{O}_{ne}$])

$$\mathcal{O}_{ne} = \{\Lambda : \Lambda(\phi) \in \mathcal{F}_S \ \forall \phi \in \mathcal{F}_S\}$$

$$\mathcal{O}_{SEP} \subsetneq \mathcal{O}_{ne}$$ → problem set

$\mathcal{O}_{ne} \hat{=} \mathcal{O}_{max}$ for general QRTs

<u>Def. (completely resource non-generating ops. [$\mathcal{O}_{c\text{-}max}$])</u>

$$\mathcal{O}_{c\text{-}max} = \{\Lambda : A \to A' : \forall \phi \in A \otimes B, \mathcal{F} \\ \Lambda \otimes id_B(\phi) \in \mathcal{F}\}$$

Resource monotones & measures

idea: quantify resource by non-negative fct. $f: \mathcal{S} \to \mathbb{R}_0^+$

monotonicity

$$f(\Omega(\varrho)) \leq f(\varrho) \quad \forall \Omega \in \mathcal{O}, \varrho \in \mathcal{S}$$

discriminance

$$f(\varphi) = 0 \Leftrightarrow \varphi \in \mathcal{F}$$ "f is faithful"

weak discriminance (WD)

$$f(\varphi) = 0 \quad \forall \varphi \in \mathcal{F}$$

Def. (resource monotone)

A fct. $f: \mathcal{S} \to \mathbb{R}_0^+$ is a resource monotone for QRT $R = (\mathcal{O}, \mathcal{F})$ iff f obeys monotonicity w.r.t. $\mathcal{O}$

Def. (resource measure)

monotone + WD

desiderata & properties

- computability
- convexity

$$f(p\rho + (1-p)\sigma) \leq p f(\rho) + (1-p) f(\sigma)$$

$$\forall \rho, \sigma \in S, \quad 0 \leq p \leq 1$$

- additivity

$$f(\rho \otimes \sigma) = f(\rho) + f(\sigma) \quad \forall \rho, \sigma \in S$$

↳ weaker variants:

sub (super) additivity

$$f(\rho \otimes \sigma) \leq (\geq) f(\rho) + f(\sigma)$$

extensivity

$$f(\sigma^{\otimes n}) = n f(\rho)$$

Examples for QRT of entanglem.

- Pure states:

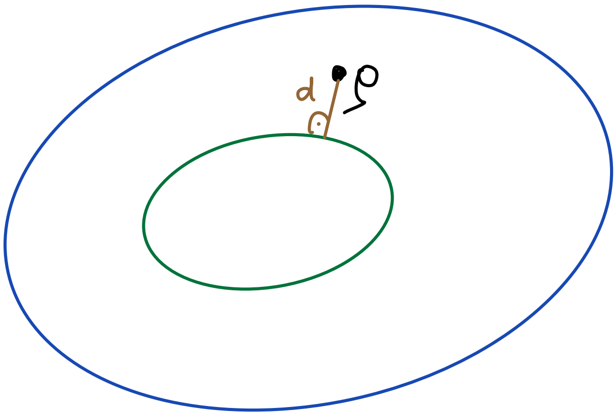
Entropy of entanglement

$$S(\mathrm{tr}_B(\psi_{AB})) = S(\mathrm{tr}_A(\psi_{AB}))$$

Schmidt decomposition:

$$|\psi_{AB}\rangle = \sum_{j=1} \lambda_j |\alpha_j\rangle |\beta_j\rangle$$

- Schmidt rank r

- 

$$E_d := \inf_{\varphi \in \mathcal{F}} d(\varrho, \varphi)$$

Examples of further ORTs

	$\mathcal{O}$	$\mathcal{F}$	monotone ↓ f
coherence w.r.t. $\{\lvert i \rangle\}$	e.g. strictly incoherent ops.	incoherent states $\phi = \sum p_i \lvert i \rangle\langle i \rvert$	relative entropy of coherence
Athermality	Gibbs-preserving ops. (GPOs) $\Gamma(\gamma_\beta) = \gamma_\beta$ thermal ops. (TOs) $\varrho' = \mathrm{tr}_B[U \varrho \otimes \gamma_\beta U^\dagger]$ with $[U, H_S + H_B] = 0$	$\gamma_\beta \propto e^{-\beta H}$	Renyi-α free energies
Entanglement	LOCC separable ops.	separable states e.g. $\phi = \sum p_j \alpha_j \otimes \beta_j$	EoE Rel. Entropy of E. ...

For almost all of these: multiple valid $\mathcal{O}$

Recommended references for further study

Entanglement Theory

Martin B Plenio and Shashank Virmani. "An introduction to entanglement measures". Quantum Information & Computation **7**, 1–51 (2007). arXiv:0504163

Ryszard Horodecki, Paweł Horodecki, Michał Horodecki, and Karol Horodecki. "Quantum entanglement". Reviews of Modern Physics **81**, 865–942 (2009)

Quantum Resource Theories

Eric Chitambar and Gilad Gour. "Quantum resource theories". Reviews of Modern Physics **91**, 25001 (2019). arXiv:1806.06107

Gilad Gour. "Resources of the Quantum World" (2024). arXiv:2402.05474

QRTs of Coherence

Alexander Streltsov, Gerardo Adesso, and Martin B. Plenio. "Colloquium: Quantum coherence as a resource". Reviews of Modern Physics **89**, 041003 (2017). arXiv:1609.02439

QRTs of Athermality

Matteo Lostaglio. "An introductory review of the resource theory approach to thermodynamics". Reports on Progress in Physics **82**, 1–31 (2019). arXiv:1807.11549

Exercises - Entanglement and Quantum Resource Theories

Problem 1: Bell states and maximal entanglement

The four Bell states form a basis for the Hilbert space of two qubits. In this exercise we will further explore why they are called 'maximally entangled'.

(i) Write down the four Bell states and consider local unitary operations of the form $U_{AB} = U_A \otimes U_B$. Can any Bell state converted into any other by such a local unitary operation? If no, provide a proof. If yes, specify a sufficient set of local unitary operations.

(ii) Can a product state $|\psi\rangle_{AB} = |a\rangle \otimes |b\rangle$ be obtained from a Bell state by a local unitary operation?

(iii) It is important to know that quantum teleportation does not permit faster than light communication. Proof this by showing that in the teleportation protocol introduced in the lecture, Bob does not obtain any information about Alice' input state $|\psi\rangle$ until she has communicated her measurement result. Why is your result is not in conflict with the functioning of quantum teleportation?

(iv) Show that Alice can use an extension of the teleportation protocol from the lecture to create any shared state $|\sigma\rangle_{AB}$ between her laboratory and Bob's by using one shared Bell state $|\phi^+\rangle$.
Hint: Consider Alice and Bob to start with the initial state $|\sigma\rangle_{12} \otimes |\phi^+\rangle_{AB}$ where Alice controls laboratories 1, 2, and A; Bob controls laboratory B. As in the original teleportation protocol encountered in the lecture, Alice and Bob may only perform measurements and unitary operations on systems in their control, as well as classical communication.

(v) We have called the Bell states 'maximally entangled'. How is this terminology justified for pure two-qubit states in light of what we have found out in this exercise?

Problem 2: QRTs from basic postulates

In the lecture, we have stated the *free operations postulate (FOP)* as a postulate. In this exercise, we will show that it can in fact be derived from other basic postulates for quantum resource theories (QRTs), namely

(free preparation of free states) The set of free states $\mathcal{F}$ is in one-to-one correspondence with the set of free preparation maps $\Omega_\phi : \mathbb{C} \to \mathcal{H}$ ($\forall\, \mathcal{H}$).
In other words, ϕ is a free state if and only if $\mathcal{O}$ includes a free map $\Omega_\phi : \mathbb{C} \to \mathcal{H}$ with $\Omega(1) = \phi$.

(free identity operations) For any permitted Hilbert space $\mathcal{H}$, the associated identity operation is a free operation: $\mathbf{id}_\mathcal{H} \in \mathcal{O}$,

(free concatenation of free operations) $\Omega, \Omega' \in \mathcal{O} \Rightarrow \Omega' \circ \Omega \in \mathcal{O}$, for all Ω and Ω' for which the output Hilbert space of Ω matches the input Hilbert space of Ω' and where $\circ$ stands for successive application.

(i) Write down the FOP. (ii) Then, derive it from the above.

Problem 3: Separable states and free operations

For an N-partite Hilbert space, recall the set of separable states $\mathcal{F}_s$ and the set of separable operations ($\mathcal{O}_{sep}$).

(i) Consider $\mathcal{O}_{c-max}$ induced by $\mathcal{F}_s$. Show that $\mathcal{O}_{sep} \subseteq \mathcal{O}_{c-max}$ – i.e., there is no operation in $\mathcal{O}_{sep}$ which is not also in $\mathcal{O}_{c-max}$.
Remark: In fact, $\mathcal{O}_{sep} = \mathcal{O}_{c-max}$ but you don't need to prove the other direction – i.e., that there is no operation in $\mathcal{O}_{c-max}$ which is not also in $\mathcal{O}_{sep}$.

(ii) Consider the case of bipartite separable states $\mathcal{F}_{2s}$. Show that any bipartite separable state can be written as a classical mixture of pure, orthonormal, bipartite product states (i.e., states of the form $|a\rangle_A |b\rangle_B$). Is any classical mixture of product states a separable state? (*Remark:* since this technically includes mixtures with a single component, note that the only separable pure states are product states).

(iii) Consider again $\mathcal{F}_{2s}$. The set $\mathcal{O}_{max}$ induced by $\mathcal{F}_{2s}$ is the set of *non-entangling operations* $\mathcal{O}_{ne}$. The swap operation $SWAP$ is implicitly defined by $SWAP(|a\rangle_A |b\rangle_B) = |b\rangle_A |a\rangle_B$ for any pair of systems and states. Recalling that $\mathcal{O}_{c-max} = \mathcal{O}_{sep}$, Show that for $\mathcal{F}_{2s}$, $\mathcal{O}_{c-max} \subsetneq \mathcal{O}_{max}$ by proofing that $SWAP \in \mathcal{O}_{ne}$ and $SWAP \notin \mathcal{O}_{sep}$.

Problem 4: Properties of entanglement monotones

In this exercise we will examine some of properties and desiderata for entanglement monotones more closely by focusing on specific examples.

(i) Show that the **Schmidt rank** is an entanglement monotone for pure states. How can we turn the Schmidt rank from a monotone into a measure (i.e., by making it weakly discriminant)? Is this derived measure faithful?

(ii) We can formally construct a **resource monotone from a distance function** $\mathbf{d}$ by defining:

$$E_d(\rho) := \inf_{\varphi \in \mathcal{F}} d(\rho, \varphi)$$

Note that any E_d defined in this way is faithful.

Show that if d obeys a data processing inequality $d(\rho, \sigma) \geq d(\Lambda(\rho), \Lambda(\sigma))$ for all quantum channels Λ (not just free ones) then E_d is monotonic under all non-entangling operations.
Remark: Recall the requirements that a distance function has to fulfill:

- $d(x, y) \geq 0$ *(non-negativity)*
- $d(x, y) = 0 \Leftrightarrow x = y$ *(identity of indiscernibles)*
- $d(x, y) = d(y, x)$ *(symmetry)*
- $d(x, z) \leq d(x, y) + d(y, z)$ *(triangle inequality)*

(iii) Recall the **relative entropy** $R(\rho||\sigma) := -S(\rho) - \mathrm{Tr}[\rho \log \sigma]$. R is not a distance on state space because it is neither symmetric $R(\rho||\sigma) \neq R(\sigma||\rho)$ nor does it satisfy the triangle inequality. This is equally true of its classical variant (called Kullback-Leibler divergence).

a) Show that R is not symmetric by finding two states, ρ and σ as an example.

Regardless of this, R can still be used as an entanglement monotone E_R using the same construction as for E_d as above.

b) The quantum relative entropy obeys joint convexity in its arguments, i.e., for $p+q = 1$ and states $\rho, \sigma, \alpha, \beta$::

$$R(p\rho + q\sigma||p\alpha + q\beta) \leq pR(\rho||\alpha) + qR(\sigma||\beta)$$

Use this to show that E_R is convex.