The Classical and Relative Frequency Definitions of Probability

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Recall:

Definition

Let $(S, \mathcal{A})$ be a sample space and a collection of events, i.e. subsets of S. A *probability* is a function P that assigns to all events a number between 0 and 1 (mathematically: $P : \mathcal{A} \to [0, 1]$) such that the two Axioms of Probability hold:

1. $P(S) = 1$,
2. $P(A_1 \cup A_2 \cup \cdots) = \sum_i P(A_i)$, whenever $A_1, A_2, \ldots$ are mutually exclusive events in $\mathcal{A}$.

- Any definition or interpretation of probability *must* satisfy these conditions.

Classical definition

Definition

The classical definition of probability assigns to the event $A \subseteq S$ the number

$$P(A) = \frac{|A|}{|S|}.$$

- Is this a probability?
- $0 \leq P(A) \leq 1$ $(A \subseteq S)$
- $P(S) = \frac{|S|}{|S|} = 1$ (1st axiom)

Second Axiom

Let $A_1, A_2, \ldots$ be mutually exclusive. Then

$$\begin{aligned}
P(A_1 \cup A_2 \cup \cdots) &= \frac{|A_1 \cup A_2 \cup \cdots|}{|S|} \\
&= \frac{|A_1| + |A_2| + \cdots}{|S|} \\
&= \frac{|A_1|}{|S|} + \frac{|A_2|}{|S|} + \cdots \\
&= \sum_i \frac{|A_i|}{|S|} \\
&= \sum_i P(A_i).
\end{aligned}$$

$\Rightarrow$ Yes, P is a probability.

- For every possible outcome $s_i \in S$, define the number

$$p_i = \frac{1}{|S|}.$$

$\Rightarrow$ this assumes that every outcome is equally likely.

- "Recover" the classical probability as follows:
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$$P(A) = \sum_{\{k \in A\}} p_k,$$

since

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$$\sum_{\{k \in A\}} p_k = \sum_{\{k \in A\}} \frac{1}{|S|} = \frac{|\{k \in A\}|}{|S|} = \frac{|A|}{|S|}.$$

Example: Roll a die

- $S = \{1, 2, 3, 4, 5, 6\}$.
- Let:
$$A = \{\text{an even number occurs}\} = \{2, 4, 6\}.$$
- Then:
$$P(A) = \frac{|A|}{|S|} = \frac{|\{2, 4, 6\}|}{|\{1, 2, 3, 4, 5, 6\}|} = \frac{3}{6} = \frac{1}{2}.$$
- Or: “all k in A” are 2, 4, and 6,
- so:
$$P(A) = \sum_{\{k \in A\}} p_k = p_2 + p_4 + p_6 = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}.$$

Relative frequency definition

Definition

Let $S = \{s_1, \ldots, s_n\}$ be a sample space and let $p_1, p_2, \ldots, p_n$ be such that

1. all p_i are in $[0, 1]$ and
2. $\sum_{i=1}^{n} p_i = 1$.

The relative frequency probability assigns to every event A the probability

$$P(A) = \sum_{\{k \in A\}} p_k.$$

- Classical definition is special case of relative frequency definition:

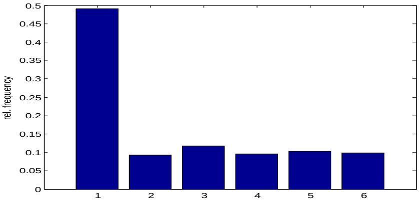
$\Rightarrow$ $p_1 = \cdots = p_n = 1/|S|$.

Is this a probability?

- $P(A) \geq 0$ (since all p's non-negative)
- $P(A) = \sum_{\{k \in A\}} p_k \leq \sum_{\{k \in S\}} p_k = \sum_{k=1}^{n} p_k = 1.$
- First axiom: $P(S) = \sum_{\{k \in S\}} p_k = \sum_{k=1}^{n} p_k = 1.$
- Second axiom: Let $A_1, A_2, \ldots$ be mutually exclusive. Then

$$\begin{aligned} P(A_1 \cup A_2 \cup \cdots) &= \sum_{\{k \in A_1 \cup A_2 \cup \cdots\}} p_k \\ &= \sum_{\{k \in A_1\} \cup \{k \in A_2\} \cup \cdots} p_k \\ &= \sum_{\{k \in A_1\}} p_k + \sum_{\{k \in A_2\}} p_k + \cdots \\ &= P(A_1) + P(A_2) + \cdots = \sum_i P(A_i). \end{aligned}$$

You rolled a die 1,000 times and observed:



Example (cont'd)

- Classical probability ($p_1 = \cdots = p_6 = 1/6$)? No!
- Relative frequencies: $p_2 = \cdots = p_6 = 1/10$.
- Since $\sum_{i=1}^{6} p_i = 1$, we must take $p_1 = 1/2$.
- Let $A = \{2, 4, 6\}$. Then

$$P(A) = \sum_{\{k \in A\}} p_k = p_2 + p_4 + p_6 = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{3}{10}.$$

- Let $B = \{1, 3, 5\}$. Then

$$P(B) = \sum_{\{k \in B\}} p_k = p_1 + p_2 + p_3 = \frac{1}{2} + \frac{1}{10} + \frac{1}{10} = \frac{7}{10}.$$