Topic 3: *MATRICES*

Jacques (3rd Edition):

Chapter 7.1- 7.2

Content

- **Adding, Subtracting and Multiplying Matrices**
- **Matrix Inversion**
- **Example: Model of National Income**

A Vector: list of numbers arranged in a row or column

e.g. consumption of 10 units X and 6 units of Y gives a consumption vector (X,Y) of (10,6) $\neq$ (6,10)

A Matrix: a two-dimensional array of numbers arranged in rows and columns

e.g. $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ a 2 X 3 matrix

- with 2 rows and 3 columns
- component a_{ij} in the matrix is in the i^{th} row and the j^{th} column

e.g. let a_{ij} be amount good j consumed by individual i

- ***columns1-3:*** represent goods X, Y& Z
- ***rows 1-2:*** represent individuals 1 & 2

Matrix of consumption

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix} = \begin{bmatrix} 0 & 10 & 5 \\ 4 & 0 & 6 \end{bmatrix}$$

Individual 1 consumes 0 of X, 10 of Y and 5 of Z
Individual 2 consumes 4 of X, 0 of Y and 6 of Z

NOTE

Row Vector is a matrix with only 1 row : A = [5 4 3] 1 X 3 matrix

Column Vector is a matrix with only 1 column : $A = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$ 3 X 1 matrix

Transposing Matrices

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$ 2 X 3 matrix

Then

$$\mathbf{A}^{\mathbf{T}} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$ 3X2 matrix

the transpose of a matrix replaces rows by columns.

$$\mathbf{A} = \begin{bmatrix} 0 & 10 & 5 \\ 4 & 0 & 6 \end{bmatrix} \text{ then } \mathbf{A}^{\mathbf{T}} = \begin{bmatrix} 0 & 4 \\ 10 & 0 \\ 5 & 6 \end{bmatrix}$$

Adding and Subtracting Matrices

Matrices **must** have same number of rows and columns, *m X n*

Just add (subtract) the corresponding elements…..

A + B + C = D i.e. $a_{ij} + b_{ij} + c_{ij} = d_{ij}$

$$\begin{bmatrix} 9 & -3 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ -1 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 2 \\ 6 & 5 \end{bmatrix}$$

A - B = E i.e. $a_{ij} - b_{ij} = e_{ij}$

$$\begin{bmatrix} 9 & -3 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 2 \\ -1 & 6 \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 5 & -5 \end{bmatrix}$$

Multiplying Matrices

To multiply A and B,

No. Columns in A = No. Rows in B

Then A x B = C
(1x 3) (3x 2) = (1x 2)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} . \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \end{bmatrix}$$

$c_{11} = (a_{11}.b_{11}) + (a_{12}.b_{21}) + (a_{13}.b_{31})$

$c_{12} = (a_{11}.b_{12}) + (a_{12}.b_{22}) + (a_{13}.b_{32})$

$$\begin{bmatrix} 2 & 3 & 4 \end{bmatrix} . \begin{bmatrix} 1 & 2 \\ 5 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 25 & 29 \end{bmatrix}$$

$c_{11} = (2x1) + (3x5) + (4x2) = 25$

$c_{12} = (2x2) + (3x3) + (4x4) = 29$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 5 & 4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 & 5 & 4 \\ 23 & 17 & 6 & 5 \end{bmatrix}$$

$c_{11} = (2x3) + (1x1) + (0x5) = 7$
$c_{12} = (2x1) + (1x0) + (0x4) = 2$
$c_{13} = (2x2) + (1x1) + (0x1) = 5$
$c_{14} = (2x1) + (1x2) + (0x1) = 4$

$c_{21} = (1x3) + (0x1) + (4x5) = 23$
$c_{22} = (1x1) + (0x0) + (4x4) = 17$
$c_{23} = (1x2) + (0x1) + (4x1) = 6$
$c_{24} = (1x1) + (0x2) + (4x1) = 5$

SCALAR MULTIPLICATION

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Then $3A = \begin{bmatrix} 3 \times a_{11} & 3 \times a_{12} \\ 3 \times a_{21} & 3 \times a_{22} \end{bmatrix}$

$A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ then $2A = \begin{bmatrix} 8 & 6 \\ 4 & 2 \end{bmatrix}$

And $3A = \begin{bmatrix} 12 & 9 \\ 6 & 3 \end{bmatrix}$

Practice Transposing, Adding, Subtracting and Multiplying Matrices using examples from any Text Book – or simply by writing down some simple matrices yourself….

Determinant of a Matrix

If A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Now we can find the determinant……

Multiply elements in *any one* row **or** *any one* column by corresponding co-factors, and sum…..

Select row 1….

$|A| = a_{11}.C_{11} + a_{12}.C_{12} = ad - bc$

Select column 2

$|A| = a_{12}.C_{12} + a_{22}.C_{22} = b(-c)+da$

MATRIX INVERSION

Square matrix: no. rows = no. columns

Identity Matrix I: AI = A and IA = A

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ (for 2 X 2 matrix)}$$

Inverse Matrix $\mathbf{A^{-1}}$: $A.A^{-1} = I \qquad A^{-1}.A = I$

TO INVERT 2 X 2 MATRIX……

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

1) Get Cofactor Matrix: $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

2) Transpose Cofactor Matrix: $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

1) multiply matrix by $\frac{1}{|A|}$ so

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ (i.e. divide each element by ad– bc)

If $|A|=0$ then there is no inverse……(matrix is singular)

Example….find the inverse of matrix A

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$|A| = ad - bc = (1.4) - (2.3) = -2$ (non-singular)

$$A^{-1} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

Check : $A.A^{-1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Example….find the inverse of matrix B

$$B = \begin{bmatrix} 2 & 4 \\ 5 & 10 \end{bmatrix}$$

$|B| = ad – bc = (2.10) – (4.5) = 0$

therefore, matrix is singular and inverse does not exist

Example Expenditure model of national income

Y = Income
C = Consumption
I = Investment
G = Government expenditure

$$Y = C+I+G \qquad (1)$$

The consumption function is

$$C = a + bY \qquad (2)$$

Note C and Y are endogenous. I and G are exogenous.

How to solve for values of endogenous variables Y and C?

Method 1

Solve the above equations directly, substituting expression for C in eq. (2) into eq. (1)

Thus, $Y = a + bY + I + G$

Solve for Y as:

$$Y - bY = a + I + G$$

$$Y(1 - b) = a + I + G$$

Thus, $Y = \dfrac{a + I + G}{1 - b}$

Substitute this value for Y into eq. (2) and solve for C:

$$C = a + b\left[\frac{a + I + G}{1 - b}\right] = \frac{(I + G)b + a}{1 - b}$$

Method 2

Now solve the same problem using matrix algebra:

- Rewrite (1) and (2) with endogenous variables, C and Y, on left hand side

From eq. 1: Y - C = I + G

From eq. 2: -bY + C = a

- Now write this in matrix notation:

$$\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} I + G \\ a \end{bmatrix}$$

or **A.X = B**

- We can solve for the endogenous variables **X,** by calculating the inverse of the **A** matrix and multiplying by **B:**

Since $\boldsymbol{AX}=\boldsymbol{B} \Rightarrow \boldsymbol{X}=\mathbf{A}^{-1}\mathbf{B}$

- To invert the 2 X 2 A matrix, recall the steps from earlier in the lecture

$$\text{If A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- In this case, where $A = \begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix}$

the determinant of A is :

$$|A| = 1.1 - [-1.-b] = 1 - b$$

Cofactor Matrix: $\begin{bmatrix} 1 & b \\ 1 & 1 \end{bmatrix}$

Transpose Cofactor Matrix: $\begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}$

The inverse is :

$$A^{-1} = \frac{1}{1-b}\begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix} \qquad = \begin{bmatrix} \frac{1}{1-b} & \frac{1}{1-b} \\ \frac{b}{1-b} & \frac{1}{1-b} \end{bmatrix}$$

- so $X = A^{-1}B$

where $X = \begin{bmatrix} Y \\ C \end{bmatrix}$ and $B = \begin{bmatrix} I + G \\ a \end{bmatrix}$

$$X = \begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} \frac{1}{1-b} & \frac{1}{1-b} \\ \frac{b}{1-b} & \frac{1}{1-b} \end{bmatrix} \begin{bmatrix} I + G \\ a \end{bmatrix}$$

Thus, multiplying $A^{-1}B$ gives,

$$\begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} \frac{I + G + a}{1-b} \\ \frac{(I+G)b + a}{1-b} \end{bmatrix}$$

These are the solutions for the endogenous variables, C and Y, just as we derived using method 1.

Method 3: Using Cramers Rule

In the example above, where

$$A = \begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} Y \\ C \end{bmatrix}$$

$$B = \begin{bmatrix} I + G \\ a \end{bmatrix}$$

- Replace column 1 of A with the elements of vector B

$$A_1 = \begin{bmatrix} I + G & -1 \\ a & 1 \end{bmatrix}$$

Calculate the determinant of this as:
$|A_1| = (I + G)(1) - (-1)(a) = I + G + a$

- We saw earlier that the determinant of A is $|A| = 1 - b$

- Therefore the solution using Cramers rule is:

$$Y = \frac{|A_1|}{|A|} = \frac{I + G + a}{1 - b}$$

- Replace column 2 of A with the elements of vector b

$$A_2 = \begin{bmatrix} 1 & I + G \\ -b & a \end{bmatrix}$$

- Calculate the determinant of this as: $|A_2| = (1)(a) - (I+G)(-b) = a + b(I+G)$

- We saw earlier that the determinant of A is

  $|A| = 1 - b$

- Therefore the solution using Cramers rule is:

$$C = \frac{|A_2|}{|A|} = \frac{a + b(I + G)}{1 - b}$$

(just as we derived using the other 2 methods)

TO INVERT 3 X 3 MATRIX……

To find inverse of 3 X 3 matrix, First need to calculate determinant

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Corresponding to each a_{ij} is a co-factor C_{ij}. 9 elements in 3X3 $\Rightarrow$ 9 co-factors.

Co-factor C_{ij} = determinant of 2X2 matrix obtained by deleting row i and column j of A, prefixed by + or – according to following pattern…

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

e.g. C_{23} is co-factor associated with a_{23}, in row 2 and column 3

so delete row 2 and column 3 to give a 2X2 matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

co-factor C_{23} is – determinant of 2X2 matrix (negative sign in position a_{23})

$$C_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -(a_{11}.a_{32} - a_{12}.a_{31})$$

e.g find all co-factors of matrix

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 3 & 7 \\ 2 & 1 & 3 \end{bmatrix}$$

C_{11} = (delete row 1 column 1, compute determinant of remaining 2X2 matrix, position a_{11} associated with +)

$$\begin{bmatrix} 2 & 4 & 1 \\ 4 & 3 & 7 \\ 2 & 1 & 3 \end{bmatrix} \quad \text{and } +\begin{vmatrix} 3 & 7 \\ 1 & 3 \end{vmatrix} = +[3.3 - (7.1)] = 2$$

C_{12} = (delete row 1 column 2, compute determinant of remaining 2X2 matrix, position a_{21} associated with -)

$$\begin{bmatrix} 2 & 4 & 1 \\ 4 & 3 & 7 \\ 2 & 1 & 3 \end{bmatrix} \quad \text{and } -\begin{vmatrix} 4 & 7 \\ 2 & 3 \end{vmatrix} = -[4.3 - (7.2)] = +2$$

Other co-factors compute as

$$C_{13} = +\begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = +[4.1 - (3.6)] = -2$$

$$C_{21} = -\begin{vmatrix} 4 & 1 \\ 1 & 3 \end{vmatrix} = -[4.3 - (1.1)] = -11$$

$$C_{22} = +\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = +[2.3 - (1.2)] = 4$$

$$C_{23} = -\begin{vmatrix} 2 & 4 \\ 2 & 1 \end{vmatrix} = -[2.1 - (4.2)] = 6$$

$$C_{31} = +\begin{vmatrix} 4 & 1 \\ 3 & 7 \end{vmatrix} = +[4.7 - (1.3)] = 25$$

$$C_{32} = -\begin{vmatrix} 2 & 1 \\ 4 & 7 \end{vmatrix} = -[2.7 - (1.4)] = -10$$

$$C_{33} = +\begin{vmatrix} 2 & 4 \\ 4 & 3 \end{vmatrix} = +[2.3 - (4.4)] = -10$$

$$\text{Co-factor Matrix} = \begin{bmatrix} 2 & 2 & -2 \\ -11 & 4 & 6 \\ 25 & -10 & -10 \end{bmatrix}$$

Now we can find the determinant......

Multiply elements in *any one* row **or** *any one* column by corresponding co-factors, and sum.....

Select row 1....

$|A| = a_{11}.C_{11} + a_{12}.C_{12} + a_{13}.C_{13}$

or equivalently select column 2

$|A| = a_{12}.C_{12} + a_{22}.C_{22} + a_{32}.C_{32}$

so the determinant of A= $\begin{bmatrix} 2 & 4 & 1 \\ 4 & 3 & 7 \\ 2 & 1 & 3 \end{bmatrix}$

(choose row 2 for example....)

$|A| = a_{21}.C_{21} + a_{22}.C_{22} + a_{23}.C_{23}$

$= (4.-11) + (3.4) + (7.6) = 10$

Now we can find the Inverse......

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Step 1 : write matrix of co-factors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 2 & 2 & -2 \\ -11 & 4 & 6 \\ 25 & -10 & -10 \end{bmatrix}$$

Step 2 : transpose that matrix (replace rows by columns), so

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} 2 & -11 & 25 \\ 2 & 4 & -10 \\ -2 & 6 & -10 \end{bmatrix}$$

Step 3: multiply each element by $\frac{1}{|A|}$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 2 & -11 & 25 \\ 2 & 4 & -10 \\ -2 & 6 & -10 \end{bmatrix}$$

$$\text{So } A^{-1} = \begin{bmatrix} \frac{1}{5} & -\frac{11}{10} & \frac{5}{2} \\ \frac{1}{5} & \frac{2}{5} & -1 \\ -\frac{1}{5} & \frac{3}{5} & -1 \end{bmatrix}$$

Check : $A.A^{-1} = I$

Practice inverting various 2X2 and 3X3 matrices using examples from Jacques, or other similar text books.

Questions Covered
Topic 3: *MATRICES*

- **Adding, Subtracting and Multiplying Matrices**
- **Matrix Inversion**
- **Example: Model of National Income**