Lecture # 14 - Optimization of Functions of One Variable (cont.)

Extreme points for concave and convex functions

- We saw last lecture that x_0 is a maximum point for $f(\cdot)$ if
  - $f'(x) \geq 0$ for $x \leq x_0$
  - $f'(x) \leq 0$ for $x \geq x_0$
- But, if a function satisfies $f'(x) \geq 0$ for $x \leq x_0$, AND then $f'(x) \leq 0$ for $x \geq x_0$, then we can say that the first order derivative is decreasing
- Recall that a function is said to be concave if $f''(x) \leq 0$, so that its first order derivative is decreasing
- Then:
  - If $f(\cdot)$ is concave, and x_0 is a stationary point for $f(\cdot)$, then x_0 is a maximum point
- Similarly, we saw that x_0 is a minimum point for $f(\cdot)$ if
  - $f'(x) \leq 0$ for $x \leq x_0$
  - $f'(x) \geq 0$ for $x \geq x_0$
- But, if a function satisfies $f'(x) \leq 0$ for $x \leq x_0$, AND then $f'(x) \geq 0$ for $x \geq x_0$, then the first order derivative is INcreasing
- Recall that a function is said to be convex if $f''(x) \geq 0$, so that its first order derivative is INcreasing
- Then:
  - If $f(\cdot)$ is convex, and x_0 is a stationary point for $f(\cdot)$, then x_0 is a minimum point

Second-Order derivative test

- More general definition (for functions that are neither concave or convex)
- Suppose $f(\cdot)$ is twice differentiable in an interval I, and suppose x_0 is an interior point of I
  - If $f'(x_0) = 0$ and $f"(x_0) < 0$, then x_0 is a (strict) maximum point
  - If $f'(x_0) = 0$ and $f"(x_0) > 0$, then x_0 is a (strict) minimum point
  - If $f'(x_0) = 0$ and $f"(x_0) = 0$, then
    - we can use the first order derivative test OR
    - use a more powerful test (see next page)

Example 1 $y = x^3 - 12x^2 + 36x + 8$

- *First order condition:* $f'(x) = 3x^2 - 24x + 36 = 0$. *The solutions to the quadratic equation are* $x = 6$ *and* $x = 2$. *So there are two stationary points.*
- *Second order condition:* $f''(x) = 6x - 24$
  - $f''(6) = 6(6) - 24 = 12 > 0 \rightarrow x = 6$ *is a minimum point*
  - $f''(2) = 6(2) - 24 = -12 > 0 \rightarrow x = 2$ *is a maximum point*

Example 2 *Suppose* $y = x^4$

- *First order condition:* $f'(x) = 4x^3 = 0$. *The unique solution is* $x = 0$, *which will be the stationary point.*
- *Second order condition:* $f''(x) = 12x^2$, *so* $f''(0) = 0$ *and the second derivative test is inconclusive.*

Example 3 *Suppose* $y = x^3$

- *First order condition:* $f'(x) = 3x^2 = 0$. *The unique solution is* $x = 0$, *which will be the stationary point.*
- *Second order condition:* $f''(x) = 6x$, *so* $f''(0) = 0$ *and the second derivative test is inconclusive.*

Nth Derivative Test

- When the second derivative, evaluated at the stationary point, is $f''(x_0) = 0$, then we need a more powerful test $\rightarrow$ the Nth Derivative Test
- The Nth Derivative Test is based on the Taylor polynomial
- Suppose $f(\cdot)$ is continuously differentiable in an interval I, and suppose x_0 is an interior point of I. Further suppose $f''(x_0) = 0$
- Suppose that the Nth derivative of $f(\cdot)$ is the first one that is NOT zero when evaluated at x_0. In math terms, $f''(x_0) = f'''(x_0) = f^{(4)}(x_0) = ... = f^{(N-1)}(x_0) = 0$, but $f^{(N)}(x_0) \neq 0$
  - If N is an even number, and $f^{(N)}(x_0) < 0$, then x_0 is a (strict) maximum point.
  - If N is an even number, and $f^{(N)}(x_0) > 0$, then x_0 is a (strict) minimum point.
  - If N is an odd number, then x_0 is an inflection point (neither a maximum nor a minimum)

Example 4 *Suppose* $y = x^4$

- *First order condition:* $f'(x) = 4x^3 = 0$. *The unique solution is* $x = 0$, *which will be the stationary point.*
- *Second derivative:* $f''(x) = 12x^2$, *so* $f''(0) = 0$.
- *Third derivative:* $f'''(x) = 24x$, *so* $f'''(0) = 0$.
- *Fourth derivative:* $f^{(4)}(x) = 24 > 0$, *so* x_0 *is a minimum point.*

Example 5 *Suppose* $y = x^3$

- *First order condition:* $f'(x) = 3x^2 = 0$. *The unique solution is* $x = 0$, *which will be the stationary point.*
- *Second order condition:* $f''(x) = 6x$, *so* $f''(0) = 0$.
- *Third derivative:* $f'''(x) = 6$, *so since* $N = 3$, x_0 *is an inflection point.*

Other critical values

- We have defined tests for local (or relative) maximum and minimum.at the interior of the domain of a particular function
- Such tests assume that
  - the function is differentiable at all points
  - the end points of the domain (or interval) are not important
- BUT, we need to consider those points as well
- So to find possible local maxima and minima for a function $f(\cdot)$ defined in an interval I, we search among the following types of points:
  - Interior points in I where $f'(x) = 0$
  - End points of I (if included in I)
  - Interior points in I where f' does not exist.

Example 6 *Consider again* $y = x^3 - 12x^2 + 36x + 8$. *We found that:*

- $x = 6$ *is a local minimum point. In fact the value of the function at* $x = 6$ *is* $f(6) = 8$
- $x = 2$ *is a local maximum point.In fact the value of the function at* $x = 2$ *is* $f(2) = 40$

The function is differentiable at all points, so there is no point at which f' *does not exist*

However, the end points may matter. Suppose the function is defined in the interval $I = [-2, 10]$. *Then:*

- *The value of the function at* $x = 10$ *is* $f(10) = 336$. *So* $x = 10$ *is also a local maximum point.*
- *The value of the function at* $x = -2$ *is* $f(-2) = -120$. *So* $x = -2$ *is also a local minimum point*

Economic Examples:

1. **Production with one input**

 - Suppose we are farmers, producing corn, and we use only one input, say labor, called L. So the production function is $Y = F(L)$
 - Suppose P is the price of corn, and w is the price of labor (i.e., the wage rate)
 - Profits are then $\Pi(L) = P \cdot F(L) - wL$
 - Firms will choose the amount of labor L^* so that profits will be maximized at the point where $\Pi'(L^*) = 0$. Such condition can be written as:

$$P \cdot F'(L^*) = w \tag{1}$$

 - Note: we will obviously need that
   - $\Pi'(L) \geq 0$ for $L \leq L^*$ AND $\Pi'(L) \leq 0$ for $L \geq L^*$
   - OR $\Pi''(L^*) = P \cdot F''(L^*) < 0$
 - Economic interpretation of first order condition (1) :
   - If we increase labor (say, by a unit), we produce $F'(L)$ more units of corn. So the left hand side is the value of additional units of corn produced when we increase labor.
   - On the right hand side, we have the cost of increasing labor, which is equal to the wage
   - If $P \cdot F'(L^*) > w$, then we should increase labor, because the gains from it exceed our losses
   - If $P \cdot F'(L^*) < w$, then we should DEcrease labor, because the gains from labor do not compensate our losses
   - So we should increase the amount of labor up to the point L^* at which our gains and losses are equal.

Example 7 *Suppose* $F(L) = \sqrt{L}$, $P = 20$ *and* $w = 1$

 - *Equation* (1) *is* $10L^{-\frac{1}{2}} = 1$ *so* $L^* = 10$.
 - *Second order condition:* $\Pi''(L^*) = 10 \cdot F''(L^*) = -5L^{-\frac{3}{2}} < 0$ *for any* $L > 0$

2. **Profit maximization**

- Suppose a profit-maximizing firm produces a single commodity
  - Total revenue is a function of its quantity produced: $R(Q)$
  - The associated total cost function is $C(Q)$
- Then profits are $\Pi(Q) = R(Q) - C(Q)$
- Suppose there is a mzimum quantity $\overline{Q}$ that the firm can produce in a given period. So the relevant interval is $\left[0, \overline{Q}\right]$
- Then the first order condition is $\Pi'(Q^*) = R'(Q^*) - C'(Q^*) = 0$, or $R'(Q^*) = C'(Q^*)$. In words, at the production level Q^* profits reach a maximum, and at such point marginal revenue equals marginal cost
- Economic interpretation
  - If $R'(Q^*) > C'(Q^*)$, then increasing production will raise our revenue by more than the raise in our cost $\Longrightarrow$ increase production
  - If $R'(Q^*) < C'(Q^*)$, then increasing production will raise our cost by more than the raise in our revenue$\Longrightarrow$ decrease production
  - So in equilibrium the marginal revenue of selling an extra unit is equal to the marginal cost of producing that unit.

- Note 1: we will obviously need that
  - $\Pi'(Q) \geq 0$ for $Q \leq Q^*$ AND $\Pi'(Q) \leq 0$ for $Q \geq Q^*$
  - OR $\Pi''(Q^*) = P \cdot F''(Q^*) < 0$
- Note 2: In special cases, it is possible that the maximum occur at $Q = 0$ or $Q = \overline{Q}$.

3. **Profit maximization of a perfectly-competitive firm**

- Suppose the firm gets a fixed price P for its product.
- Then $R(Q) = PQ$, so $R'(Q) = P$. In words, when a firm takes price as given, marginal revenue equals price
- Then the first order condition takes the form: $P = C'(Q^*)$

Example 8 *Suppose* $P = 80$ *and* $C(Q) = 100 + 10Q + \frac{1}{2}Q^2$. *Also suppose* $\overline{Q} = 100$

Then $R'(Q) = P = 80$. *And* $C'(Q) = 10 + Q$. *So* $Q^* = 70$

Then $\Pi(80) = (80)(70) - C(70) = 5600 - 3250 = 2350$

Notice also that $\Pi(0) = 0 - C(0) = -100$, *while* $\Pi\left(\overline{Q}\right) = (80)(100) - C(100) = 1900$

Example 9 *Suppose now that* $P = 120$

Then $R'(Q) = P = 120$. *And* $C'(Q) = 10 + Q$. *So* $Q^* = 110$

But $Q^* = 110$ *is outside the range, so we must look at the end points*

$\Pi(0) = 0 - C(0) = -100$, *while* $\Pi\left(\overline{Q}\right) = (120)(100) - C(100) = 12000 - 6100 = 5900$

So the firm will produce at $\overline{Q} = 100$

4. **Profit maximization of a monopolist firm**

- Suppose the monopolist faces an inverse demand $P(Q)$
- Then $R(Q) = P(Q) \cdot Q$, and $R'(Q) = P'(Q) \cdot Q + P(Q)$
  - Notice that, since $P'(Q) < 0$, then $R'(Q) < P(Q)$.
- Then the first order condition becomes: $P'(Q) \cdot Q + P(Q) = C'(Q^*)$

Example 10 *Suppose* $P = 110 - 2Q$

Then $R'(Q) = P'(Q) \cdot Q + P(Q) = (-2)Q + 110 - 2Q = 110 - 4Q$

As before, $C'(Q) = 10 + Q$.

So $110-4Q = 10+Q.$, *and the solution is* $Q^* = 20$, *and the price is* $P = 80-2(20) = 40$

Then $\Pi(20) = (40)(20) - C(20) = 800 - 500 = 300$

Notice also that $\Pi(0) = 0 - C(0) = -100$, *while* $\Pi(\overline{Q})$ *cannot be defiined because* $P(100) = 110 - 2(100) = -90 < 0$