Assignment 1

Question 1

a) The autocorrelation at lag k , ρ_k, of a stationary process is defined as:

$$\rho_k = corr(y_t, y_{t-k}) = \frac{cov(y_t, y_{t-k})}{var(y_t)}$$

for each value of $k = 1, 2, ...$

The autocorrelations expressed as a function of k are called *autocorrelation function.*

However, we only have a realization of a stochastic process, therefore we can only compute the sample autocorrelation function. Assume to have T observations.

Indicate with $\overline{y}$ the sample mean:

$$\overline{y} = \frac{1}{T}\sum_{t=1}^{T} y_t$$

The sample variance is given by:

$$\widehat{\gamma}_0 = \frac{1}{T}\sum_{t=0}^{T}(y_t - \overline{y})^2$$

The sample autocorrelation is given by:

$$r_k = \frac{\frac{1}{T-k}\sum\limits_{t=k+1}^{T}(y_t - \overline{y})(y_{t-k} - \overline{y})}{\frac{1}{T}\sum\limits_{t=0}^{T}(y_t - \overline{y})^2}$$

for each value of $k = 1, 2, ...$

The sample autocorrelation function reported in the exercise shows a pattern which decreases as the lag increases. Given the properties of the ACF of MA and AR processes, this pattern could indicate that an autoregressive representation is more appropriate.

b) The partial autocorrelation between y_t and y_{t-k} measures the correlation between these two observations by eliminating the effects of the intervening values (y_{t-1} to y_{t-k-1}).

The first sample partial autocorrelation coefficient is equal to the first sample autocorrelation coefficient because there is no other intervening value between them.

Given the properties of the PACF of MA and AR processes, this pattern could indicate that an autoregressive representation is more appropriate. Besides, we could also guess the number of lags. It seems that an AR(1) is more appropriate.

Question 2

a)

$$\begin{aligned} y_t &= \vartheta y_{t-1} + \varepsilon_t \\ y_t &= \vartheta L y_t + \varepsilon_t \\ (1 - \vartheta L) y_t &= \varepsilon_t \end{aligned}$$

Note that $(1 - \vartheta L)$ is invertible if and only if $|\vartheta| < 1$. As we are assuming that the AR(1) is stationary, we can write:

$$y_t = (1 - \vartheta L)^{-1} \varepsilon_t$$

where $(1 - \vartheta L)^{-1} = 1 + \vartheta L + \vartheta^2 L^2 + \vartheta^3 L^3 + ...$

$$\begin{aligned} y_t &= (1 + \vartheta L + \vartheta^2 L^2 + \vartheta^3 L^3 + ...)\varepsilon_t \\ y_t &= \sum_{j=0}^{\infty} \varepsilon_{t-j} \end{aligned}$$

b)

$$\begin{aligned} y_t &= \phi \varepsilon_{t-1} + \varepsilon_t \\ y_t &= \phi L \varepsilon_t + \varepsilon_t \\ y_t &= (1 + \phi L)\varepsilon_t \\ y_t &= (1 - (-\phi L))\varepsilon_t \end{aligned}$$

$(1 - (-\phi L))$ is invertible if and only if $|\phi| < 1$. We are indeed assuming that this is the case, therefore we can write:

$$(1 - (-\phi L))^{-1} y_t = \varepsilon_t$$

where $(1 - (-\phi L))^{-1} = 1 + (-\phi L) + \phi^2 L^2 + (-\phi^3 L^3)... = \sum_{j=0}^{\infty} (-\phi)^j L^j$

$$\left[\sum_{j=0}^{\infty} (-\phi)^j L^j \right] y_t = \varepsilon_t$$

Question 3

1.

$$y_t = \phi y_{t-1} + \varepsilon_t$$

Iterate

$$\begin{aligned}
y_t &= \phi(\phi y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\
y_t &= \phi^2 y_{t-2} + \phi\varepsilon_{t-1} + \varepsilon_t \\
y_t &= \phi^2(\phi y_{t-3} + \varepsilon_{t-2}) + \phi\varepsilon_{t-1} + \varepsilon_t \\
y_t &= \phi^3 y_{t-3} + \phi^2\varepsilon_{t-2} + \phi\varepsilon_{t-1} + \varepsilon_t \\
&\quad ... \\
y_t &= \phi^t y_0 + \sum_{j=0}^{t-1} \phi^j \varepsilon_{t-j}
\end{aligned}$$

Note that if $|\phi| < 1$, then $\phi^t \to 0$ when $t \to \infty$.

$$\lim_{t\to\infty} y_t = \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}$$

For suffienciently large values of t:

$$\begin{aligned}
E(y_t) &= E\left(\sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}\right) \\
E(y_t) &= 0
\end{aligned}$$

$$\begin{aligned}
cov(y_t, y_{t-j}) &= E(y_t y_{t-j}) - E(y_t)E(y_{t-j}) \\
&= E(y_t y_{t-j}) \\
&= E\left[\left(\sum_{i=0}^{\infty} \phi^i \varepsilon_{t-i}\right)\left(\sum_{i=0}^{\infty} \phi^i \varepsilon_{t-j-i}\right)\right] \\
&= E\left[\left(\varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + ...\right)\left(\varepsilon_{t-j} + \phi\varepsilon_{t-j-1} + \phi^2\varepsilon_{t-j-2} + ...\right)\right] \\
&= \phi^j \sigma^2 \left[1 + \phi^2 + \phi^4 + ...\right] \\
&= \frac{\phi^j \sigma^2}{1 - \phi}
\end{aligned}$$

Let's calculate the variance:

$$
\begin{aligned}
var(y_t) &= E(y_t^2) = E\left[\left(\sum_{j=0}^{\infty}\phi^j \varepsilon_{t-j}\right)^2\right] \\
&= \sigma^2(1 + \phi^2 + \phi^4 + ...) \\
&= \sigma^2/(1 - \phi^2)
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
cov(y_t, y_{t-j}) &= \phi^j E(y_{t-j}^2) \\
&= \sigma^2 \phi^j/(1 - \phi^2)
\end{aligned}
$$

2.

$$
\begin{aligned}
corr(y_t, y_{t-j}) &= \frac{cov(y_t, y_{t-j})}{\sqrt{var(y_t) var(y_{t-j})}} \\
&= \frac{cov(y_t, y_{t-j})}{var(y_t)} \\
&= \phi^j
\end{aligned}
$$

3. Consider a stationary AR(2) process:

$$Y_t = \delta + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t$$

and take the covariance between Y_t and Y_{t-k}.

$$cov(Y_t, Y_{t-k}) = \phi_1 cov(Y_{t-1}, Y_{t-k}) + \phi_2 cov(Y_{t-2}, Y_{t-k}) + cov(\varepsilon_t, Y_{t-k})$$

Consider $k = 0, 1, 2$.

$$
\begin{aligned}
cov(Y_t, Y_t) &= \gamma_0 = \phi_1 cov(Y_{t-1}, Y_t) + \phi_2 cov(Y_{t-2}, Y_t) + cov(\varepsilon_t, Y_t) \\
cov(Y_t, Y_t) &= \gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2 + cov(\varepsilon_t, (\delta + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t)) \\
cov(Y_t, Y_t) &= \gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2 + \sigma^2 \\
cov(Y_t, Y_{t-1}) &= \gamma_1 = \phi_1 cov(Y_{t-1}, Y_{t-1}) + \phi_2 cov(Y_{t-2}, Y_{t-1}) + cov(\varepsilon_t, Y_{t-1}) \\
cov(Y_t, Y_{t-1}) &= \gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_1 \\
cov(Y_t, Y_{t-2}) &= \gamma_2 = \phi_1 cov(Y_{t-1}, Y_{t-2}) + \phi_2 cov(Y_{t-2}, Y_{t-2}) + cov(\varepsilon_t, Y_{t-2}) \\
cov(Y_t, Y_{t-2}) &= \gamma_2 = \phi_1\gamma_1 + \phi_2\gamma_0
\end{aligned}
$$

These are the Yule-Walker equations: the covariance is expressed in terms of the model parameters.

Question 4

Covariance stationary process:

1. $E(x_t) = \mu < \infty \ \forall t$
2. $var(x_t) = \gamma_0 \ \forall t$
3. $cov(x_t, x_{t-k}) = \gamma_k$

$$\begin{aligned} Y_t &= \alpha + Y_{t-1} + \varepsilon_t \\ Y_t &= \alpha + \alpha + Y_{t-2} + \varepsilon_{t-1} + \varepsilon_t \\ Y_t &= 2\alpha + \alpha + Y_{t-3} + \varepsilon_{t-2} + \varepsilon_{t-1} + \varepsilon_t \\ & \quad ... \end{aligned}$$

Suppose that the initial value Y_0 is known.

$$Y_t = t\alpha + Y_0 + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + ... + \varepsilon_1$$

Consider $E(Y_t)$

$$E(Y_t) = E(t\alpha + Y_0 + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + ... + \varepsilon_1)$$

Remember that $E(\varepsilon_t) = 0$, $\forall t$

$$E(Y_t) = t\alpha + Y_0$$

Note that $E(Y_t)$ depends on time, therefore $\{Y_t\}$ cannot be a covariance stationary process.

Question 5

1. False

   Consider the MA(1)

$$y_t = \phi\varepsilon_{t-1} + \varepsilon_t$$

where $\{\varepsilon_t\}$ is a white noise stochastic process, $\varepsilon_t \tilde{} IID(0, \sigma^2)$

1. (a)

$$E(y_t) = 0$$

   i. The mean is time independent and finite

(b)

$$
\begin{aligned}
var(y_t) &= E(y_t^2) \\
&= E\left[\left(\phi\varepsilon_{t-1}+\varepsilon_t\right)\left(\phi\varepsilon_{t-1}+\varepsilon_t\right)\right] \\
&= E\left[\phi^2\varepsilon_{t-1}^2+2\phi\varepsilon_t\varepsilon_{t-1}+\varepsilon_t^2\right)] \\
&= \phi^2 E(\varepsilon_{t-1}^2)+E(\varepsilon_t^2) \\
&= (1+\phi^2)\sigma^2
\end{aligned}
$$

i. The variance is time independent and constant

(c)

$$
\begin{aligned}
cov(y_t, y_{t-1}) &= E(y_t y_{t-1}) \\
&= E(\left(\phi\varepsilon_{t-1}+\varepsilon_t\right)\left(\phi\varepsilon_{t-2}+\varepsilon_{t-1}\right)) \\
&= \phi E(\varepsilon_{t-1}^2) \\
&= \phi\sigma^2
\end{aligned}
$$

i. and

$$cov(y_t, y_{t-j}) = 0 \ \forall j \neq 0$$

The autocovariance is time independent and finite
Any finite order MA is covariance stationary

2. True

Consider an ARMA(p,q) process:

$$y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + ... + a_p y_{t-p} + x_t$$

where $x_t = \varepsilon_t + b_1\varepsilon_{t-1} + b_2\varepsilon_{t-2} + ... + b_q\varepsilon_{t-q}$ and $\{\varepsilon_t\}$ is a white noise stochastic process, $\varepsilon_t \tilde{} IID(0, \sigma^2)$.

If the roots of the inverse characteristic equation lie outside of the unit circle then the $\{y_t\}$ is stationary.

$$y_t = \frac{a_0}{1-\sum\limits_{i=1}^{p} a_i} + \frac{\varepsilon_t}{1-\sum\limits_{i=1}^{p} a_i L^i} + \frac{b_1\varepsilon_{t-1}}{1-\sum\limits_{i=1}^{p} a_i L^i} + \frac{b_2\varepsilon_{t-2}}{1-\sum\limits_{i=1}^{p} a_i L^i} + ... + \frac{b_q\varepsilon_{t-q}}{1-\sum\limits_{i=1}^{p} a_i L^i}$$

$\{y_t\}$ sequence is stationary as long as the roots of $1-\sum\limits_{i=1}^{p} a_i L^i$ are outside the unit circle.

Question 6

See Enders, Chapter 1, Section 9

Question 7

Consider the moment matrix $E(\mathbf{x}_t\mathbf{x}_t')$ where $\mathbf{x}_t = (1, y_{t-1})'$.

$$\begin{bmatrix} 1 \\ y_{t-1} \end{bmatrix} \begin{bmatrix} 1 & y_{t-1} \end{bmatrix} = \begin{bmatrix} 1 & y_{t-1} \\ y_{t-1} & y_{t-1}^2 \end{bmatrix}$$

Take the expected value:

$$E \begin{bmatrix} 1 & y_{t-1} \\ y_{t-1} & y_{t-1}^2 \end{bmatrix} = \mathbf{\Sigma}_{xx} = \begin{bmatrix} 1 & \mu \\ \mu & \gamma_0 + \mu^2 \end{bmatrix}$$

as $var(y_{t-1}) = \gamma_0 = E\left[y_{t-1} - \mu\right]^2 = E(y_{t-1}^2) - \left(E\left(y_{t-1}\right)\right)^2$. Therefore: $E(y_{t-1}^2) = \gamma_0 + \mu^2$

Calculate the determinant of $E(\mathbf{x}_t\mathbf{x}_t')$:

$$\det \mathbf{\Sigma}_{xx} = \gamma_0 > 0$$

We can conclude that $\mathbf{\Sigma}_{xx}$ is nonsingular and hence finite.

Consider now the sample cross moment of the regressors, $\mathbf{S}_{xx} = \frac{1}{T}\mathbf{X}'\mathbf{X} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t'$.

By the Ergodic Theorem, we know that

$$\lim_{T\to\infty} \mathbf{S}_{xx} = \mathbf{\Sigma}_{xx}$$

Therefore, we can conclude that for T sufficiently large the sample cross moment of the regressors, $\mathbf{S}_{xx} = \frac{1}{T}X'X$ is nonsingular as well. Since $\frac{1}{T}X'X$ is nonsingular iff $rank(X) = K$, then the assumption of multicollinearity is satisfied with probability 1 for T sufficiently large.