MSc Econometrics II

Problem set 4

To hand in by March 12th.

Exercise 1

Consider a first-order autoregressive process, $AR(1)$

$$y_t = \vartheta_0 + \vartheta_1 y_{t-1} + \varepsilon_t$$

where $|\vartheta_1| < 1$ and $\{\varepsilon_t\}$ is a white noise stochastic process, $\varepsilon_t \sim IID(0, \sigma^2)$ and y_0 is known.

Assume that you know the actual data generating process and the past realization of $\{\varepsilon_t\}$ and $\{y_t\}$

1. Derive the j-step ahead forecast error
2. Calculate the mean and variance of the j-step ahead forecast error.

Exercise 2

Consider the autoregressive distributed lag, ADL, model given by

$$y_t = \theta_0 + \theta_1 y_{t-1} + \gamma_0 x_t + \gamma_1 x_{t-1} + \varepsilon_t$$

Derive the corresponding error correction model (ECM), and explain how it is related to co-integration.

Exercise 3

Consider the following Vector Autoregression in structural form

$$\begin{aligned} y_t &= b_{10} - b_{12} z_t + \gamma_{11} y_{t-1} + \gamma_{12} z_{t-1} + \varepsilon_{yt} \\ z_t &= b_{20} - b_{22} z_t + \gamma_{21} y_{t-1} + \gamma_{22} z_{t-1} + \varepsilon_{zt} \end{aligned}$$

where $\{\varepsilon_{yt}\}$ and $\{\varepsilon_{zt}\}$ are white noise processes.

1. Write the VAR in standard form. Show that the error terms of the VAR in standard form are white noise processes.
2. Using the VAR in standard form, derive the condition for stability.
3. Suppose the stability condition is met, derive the VAR in the Vector Moving Average (VMA) form.
4. Derive the impulse response functions. Discuss the meaning of the impulse response functions.

Exercise 4

Consider the following VAR in standard form

$$
\begin{aligned}
y_t &= \gamma_{11} y_{t-1} + \gamma_{12} z_{t-1} + e_{yt} \\
z_t &= \gamma_{21} y_{t-1} + \gamma_{22} z_{t-1} + e_{zt}
\end{aligned}
$$

where $y_t \sim I(1)$ and $z_t \sim I(1)$

1. Explain under what conditions the two processes are cointegrated.
2. How would you test for cointegration?

Exercise 5

Consider the following system of linear equations

$$
\begin{aligned}
\mathbf{y}_1 &= \mathbf{X}_1 \boldsymbol{\beta}_1 + \boldsymbol{\varepsilon}_1 \\
\mathbf{y}_2 &= \mathbf{X}_2 \boldsymbol{\beta}_2 + \boldsymbol{\varepsilon}_2 \\
& \quad .... \\
\mathbf{y}_M &= \mathbf{X}_M \boldsymbol{\beta}_M + \boldsymbol{\varepsilon}_M
\end{aligned}
$$

where $\mathbf{y}_i$, $i = 1, 2$, is a vector of observations on the *ith* dependent variable, $\mathbf{X}_i, i = 1, 2, ..M$, is a matrix of observations on k_i independent variables. Assume strict exogeneity of X_i, $E(\boldsymbol{\varepsilon}|\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_M) = \mathbf{0}$, homoscedasticity ,$E(\boldsymbol{\varepsilon}_m \boldsymbol{\varepsilon}_m'|\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_M) = \sigma_{mm} I_T$. Assume that the disturbances are uncorrelated across observations, but correlated across equations:

$$
E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'|\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_M) = \Omega = \begin{bmatrix} \sigma_{11} I & \sigma_{12} I & ... & ... & \sigma_{1M} I \\ \sigma_{21} I & \sigma_{22} I & ... & ... & \sigma_{2M} I \\ . & . & . & . & . \\ . & . & . & . & . \\ \sigma_{M1} I & \sigma_{M2} I & ... & ... & \sigma_{MM} I \end{bmatrix}
$$

1. Assume that Ω is known. *Derive* the GLS estimator.
2. Show that the GLS estimator is identical to the OLS estimator in the case of SUR with identical regressors.

Exercise 6

In the following two-equation regression model

$$
\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}
$$

where $\mathbf{y}_i$, $i = 1, 2$, is a vector of observations on the ith dependent variable, $\mathbf{X}_i, i = 1, 2$, is a matrix of observations on k_i independent variables, $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ are vectors of coefficients and the $\mathbf{u}_1$ and $\mathbf{u}_2$ are vectors of disturbances for which $E(\mathbf{u}_1) = E(\mathbf{u}_2) = 0$ and $E(\mathbf{u}_1\mathbf{u}_1') = \sigma_{11}I_T$, $E(\mathbf{u}_2\mathbf{u}_2') = \sigma_{22}I_T$ and $E(\mathbf{u}_1\mathbf{u}_2') = \sigma_{12}I_T$. It is assumed that the matrix

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$$

is positive definite and it is known.

1. Find the covariance matrix of the single equation least squares estimator of $\boldsymbol{\beta}_1$.

2. Find the covariance matrix of the Generalized Least Square (GLS) estimator of $\boldsymbol{\beta}_1$.