# CAPM

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# Reading

- RB Chs. 5, 6, and 9.1-9.4, EGBG Chs. 13-15, and CN Ch. 5, 7, and 8.
- Jagannathan, R., McGratten, E., 1995. The CAPM Debate, Quarterly Review of the Federal Reserve Bank of Minneapolis, 2-17.
- Fama, E., French, K., 2004. The Capital Asset Pricing Model: Theory and Evidence, Journal of Economic Perspectives 18, 25–46.
- Campbell, J., Vuolteenaho, T., 2004. Bad Beta, Good Beta, American Economic Review 94, 1249-1275.

# Outline

- Mean-variance model and derivation of market Beta
- Fama and Macbeth (1973) approach to testing CAPM
- Size effects
- Good beta, bad beta

# Portfolio Diversification

- Assume an individual has access to two types of asset.

1. Risk-free asset (in which he is free to borrow/lend) such as bonds (bonds are the new thing).
2. A set of risky asset(s) such as stocks

- What do investors want. A high expected return, without too much risk.
- How can they get that. Hold a diversified portfolio of assets.
- What does that mean. Find another asset not correlated with our current holdings. The point is, even if that asset has lower expected return, it can help to decrease variance on the return of the portfolio.

# Example of Two Risky Assets

- Two risky assets have the same expected return and variance. For assets 1 and 2, say,

$$E(R_1) = E(R_2)$$

$$\sigma_1 = \sigma_2$$

- Assume they are perfectly negatively correlated. If $R_1$increases by 1%, then $R_2$ decreases by 1%. That is,

$$\rho_{12} = Cor(R_1, R_2) = -1$$

- What should do we do as an investor? Allocate 50% of our wealth to each asset.
- Diversification decreases the risk on the portfolio to zero, even though expected return is independent of which asset we hold.

# General Points

- More general points on diversification:

1. If $Cor\left(R_1, R_2\right) \geqslant 0$, but not equal to 1, it still pays to diversify asset holdings.
2. Given unlimited lending/borrowing at $r$ (the risk free rate), an investor will always hold some of each asset.

- Underlying this there is an arbitrage-type argument.

1. Suppose one stock is not desired by an investor $\Rightarrow$ it's price falls (now) $\Rightarrow \mathbb{E}\left(R_{t+1}\right)$ increases, all else equal.
2. Therefore, $P_t$ decreases until $\mathbb{E}\left(R_{t+1}\right)$ increases such that investors buy the stock.

# Digression on Notation

- When considering two assets, we use the following notation.

$$R_i \text{ for } i = 1, 2 \quad \text{actual return of assets 1 and 2}$$

$$\mu_i = E\left(R_i\right) \quad \text{expected return}$$

$$\sigma_i^2 = E\left(R_i - \mu_i\right)^2 \quad \text{variance of return}$$

$$\sigma_{12} = E\left[\left(R_1 - \mu_1\right)\left(R_2 - \mu_2\right)\right] = Cov\left(R_1, R_2\right) \quad \text{covariance}$$

$$\rho = \frac{\sigma_{12}}{\sigma_1 \sigma_2}; \quad \rho \in \left(-1; 1\right); \quad \text{where } \rho = 0 \Rightarrow \text{not linearly related}$$

# Mean-Variance and CAPM

- Mean variance model simply says that investors objective function is a combination of the mean return and the variance of the return.
- The CAPM is market version of the mean-variance (MV) model.
- Now assume an investor has two portfolios of assets, $A$ and $B$, such that,

$$E\left(R_A\right) \geqslant E\left(R_B\right)$$
$$Var\left(R_A\right) \leqslant Var\left(R_B\right)$$

- According to the 'MV criteria', portfolio $A$ is preferred to portfolio $B$ (why? as it has a higher expected return with less variance).
- If this condition is satisfied, we have "efficient portfolios".

# Details on CAPM

- Suppose now there are two risky assets (not "portfolios of.."), 1 and 2.
- We minimize portfolio's return variance ($\sigma^2$) by allocating fractions of wealth, $\omega_1$ and $\omega_2$ ($\omega_1 + \omega_2 = 1$).
- That is, the portfolio's actual return, given an investment position, is,

$$R_p = \omega_1 R_1 + \omega_2 R_2$$

- Using the notation above, the expected return and variance of the portfolio is the following.

$$\begin{aligned} E\left(R_\rho\right) &= \omega_1 \mu_1 + \omega_2 \mu_2 \\ \sigma_p^2 &= \omega_1^2 \sigma_1^2 + \omega_2^2 \sigma_2^2 + 2\omega_1 \omega_2 \sigma_1 \sigma_2 \rho \end{aligned}$$

- Why? Recall, $R_1$ and $R_2$ are random variables.

# Details on CAPM

- To minimize the variance, set $\partial\sigma_p^2/\partial w_i = 0$.
- Rearranging,

$$w_1 = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$

- This means we can determine the amount of total wealth allocated to asset one simple through the variance-covariance structure of returns.
- Again note, if $\rho = -1$ we can diversify all risk ($\Leftrightarrow \sigma_{12} = -\sigma_1\sigma_2$).

# Results from Basic CAPM Model

- Generalizations:

1. "$N$" assets $\Rightarrow$ efficiency frontier.
2. Allow a risk-free asset with unlimited borrowing/lending (we'll do this next).

- Relation between MV model and CAPM:

1. In the former, agents choose optimal portfolios, whereas the latter is an equilibrium model of expected returns.
2. If all investors behave according to MV objective and have "equal beliefs", then what can be inferred about asset returns also explains the risk-premium for each asset.

# One way to derive market beta

- Assets and returns.

$$R_p = \sum_{i=0}^{2} w_i R_i \text{ where } \quad i = 0, 1, 2 \ \text{ and } \ i = 0 \ \text{ risk-free asset}$$

- Investor's problem.

$$\max_{w_i} U \quad s.t. \sum_{i=0}^{2} w_i = 1 \text{ where } U \left( \underbrace{E\left(R_p\right)}_{+}, \underbrace{Var\left(R_p\right)}_{-} \right)$$

- Note,

$$R_p = \sum_{i=0}^{2} w_i R_i \text{ and } E\left(R_p\right) = \sum_{i=0}^{2} w_i E\left(R_i\right) \text{ and } Var\left(R_0\right) = 0$$

$$Var\left(R_p\right) = w_1^2 \cdot Var\left(R_1\right) + w_2^2 \cdot Var\left(R_2\right) + 2 w_1 w_2 \cdot Cov\left(R_1, R_2\right)$$

# The Main Result

- The first-order condition for this problem is the following,

$$[E(R_p) - R_0] U_{R,1} + 2Var(R_p) U_{R,2} = 0$$

- Algebra steps ... skipped... you did this in term 1 ...

$$E(R_i) - R_0 = \underbrace{\left[\frac{Cov(R_i, R_p)}{Var(R_P)}\right]}_{\text{“beta”}} \underbrace{(E(R_p) - R_0)}_{\text{market risk}} = \beta_i (E(R_p) - R_0)$$

$$i = 1, 2 (, .., n)$$

- Here, $\beta$ is the market beta, $R_p$ is the return on the portfolio (say, SP500), $R_0$ is the return on the risk free asset (risk free rate, say a T-bill).

# Properties/Results of CAPM

- The portfolio is on the MV frontier of the return/variance relationship. If not, we can decrease variance (for some $\mathbb{E}(R)$) through $w_0$, $w_i$ $(i = 1, 2)$.
- The portfolio is on efficient part of the frontier if, in addition, no other portfolio has higher $E(R)$.
- CAPM implies everyone in the economy holds equal amount of risky assets $\Rightarrow$ betas computed with regard to all individual portfolios are the same.
- For every asset, the "risk premium/beta" is the same. That is, every investment opportunity provides equal compensation for any given level of risk, when scaled by the beta.

# Excess Returns

- In terms of excess returns, all we are saying is the following.

$$\underbrace{E\left(Z_i\right)}_{\text{asset}} = \beta_i \underbrace{E\left(Z_m\right)}_{\text{market}} \text{for } i = 1, 2\left(, .., n\right)$$

- So, $Z_m$ is the excess return on the market portfolio.
- Empirical tests of CAPM focus on three implications of this equation. Specifically,

1. Zero intercept
2. Beta captures all the cross sectional variation of expected excess returns
3. A positive market risk premium

- Important: think of this as being a linear regression model.

# CAPM and Cross Section Tests

- The easiest thing to do is to plot average returns against market beta. But that is like saying we want to regress a cross-section of average asset returns on estimates of assets' betas.
- Cross section points to note:

1. Expected returns (on all assets) are linearly related to their betas.
2. Positive beta-premium says that the expected return on market portfolio is greater than the expected return on assets whose returns are uncorrelated with market return.
3. Sharpe-Linter model says that assets uncorrelated with the market have expected returns equal to $r$, where beta premium is expected market return minus $r$.

# Cross Section

- The basic result from the data is that there is positive correlation between beta and the average return, but it is too flat to be consistent with the model.
- The model implies intercept equal to $r$ and estimated beta is $\mathbb{E}(R_p) - r$ , i.e. excess return. But usually the intercept is above the risk free rate and the estimated beta is less than the excess return.
- Obvious drawbacks of our approach:

1. Estimates of beta for individual assets are imprecise $\Rightarrow$ measurement error in explaining average returns.
2. Regression residuals have common sources of variation, e.g. industry effects in average returns.

# Individual Assest versus Portfolios

- We can avoid the problems in the test proposed above by using portfolios, not individual returns.
- Why? Expected returns and market betas combine in the same way in portfolios. Therefore, if CAPM explains individual security returns, it explains portfolio returns.
- Estimated betas are also more precise $\Rightarrow$ decrease errors $\Rightarrow$ better approach. The idea is that one security could have different measured beta, but the noise is high for individual beta.
- New drawback: grouping reduces the power of test.
- Possible remedy: sort securities on beta when forming portfolios $\Rightarrow$ first part: lowest betas; last part: highest betas. **This is now the standard approach.**

# CAPM and Ice-cream

- We have the following for the CLRM:

$$\underset{(n\times 1)}{Z_{i,t}} = \underset{(n\times 1)}{\alpha} + \underset{(n\times 1)}{\beta} Z_{m,t} + \varepsilon_t \quad \text{for } i = 1,2\,(,..,n)$$

$$\mathbb{E}_t\left(\varepsilon_t\right) = 0$$

$$cov\left(Z_{m,t}, \varepsilon_t\right) = 0$$

- We also need a return for the portfolio in period $t$ and a risk free return. Our null hypothesis is that $\alpha = 0$.
- This is like the ice-creams example in lecture 1.

# Fama and Macbeth (1973, JPE) Appraoch

- Another approach is to follow Fama and Macbeth's (1973, JPE) analysis. With this we attempt to see whether beta captures all the cross sectional variation of expected excess returns and if there is a positive market risk premium
- FM estimate month by month cross sections on monthly returns (i.e., cross-section regressions) over time, $t = 1, ..., T$.
- The the use the time series to estimate n monthly slopes and intercepts to test if average returns on assets is uncorrelated with average return.
- How does it work?

1. Do not estimate correlations directly.
2. Capture the effects of residual correlation of variation in regression coefficients.

# Fama and Macbeth (1973, JPE) Appraoch

- Run

$$\underset{(n\times1)}{Z_{i,t}} = \underset{(n\times1)}{\alpha} + \underset{(n\times1)}{\beta} Z_{m,t} + \varepsilon_t \quad \text{for } i = 1, 2\,(, .., n)$$

- Use, say, 60 months of data. This gives use 60 estimates of $\alpha$ and $\beta$.
- We then analyze the time series properties of $\widehat{\alpha}_t$ and $\widehat{\beta}_t$ for our 60 generated observations. The prediction are:

1. $\widehat{\alpha}_t = 0$, zero intercept.
2. $\widehat{\beta}_t > 0$, beta premium.

- True betas are unknown, however, so there are problems. We can use IV, or we can group assets into portfolios (we need a way to group assets, however).

# Grouping Assets into Portfolios

- Say we have 2000 stocks. We want to calculate a set of 100 portfolio betas to which individual stocks are assigned.
- For each stock, estimate $\beta$ from $Z_{i,t} = \alpha + \beta Z_{m,t} + \varepsilon_t$ using $t = 1 - 60$, monthly observation. Get $\widehat{\beta}_i$.
- At $t = 61$, form 10 portfolio betas based on market size, then subdivide by 10 (i.e. 100!) according to estimated $\beta_i$ and calculate average return (monthly) on these 100 portfolios.
- Over next year ($t = 61...72$) $\Rightarrow$ 100 average returns, $\overline{R_p}$ for $p = 1....10$ sorted by size, and $\beta$. Then repeat for each year and take average betas for each of 100 (sorted) portfolios.
- In each year, individual stocks are assigned a portfolio beta based on a sorted "size-beta" portfolio to which they belong (these can change over time.)

# Fama and Macbeth (1973, JPE) Appraoch

- After we have done all the hard work, we use the 100 portfolio beta in the following regression for the 2000 stocks.

$$\overline{R_i} = \lambda_0 + \lambda_1 \widehat{\beta_{pi}} + \gamma z_i + v_i$$

$$z_i \text{ is a CS company variable}$$

$$\lambda_0 = \overline{r} \text{ and } \lambda_1 = \overline{R_m} - \overline{r} > 0$$

- Repeat the CS regression for $t = 1, ...., T$ months, giving a times series for $\{\lambda_0, \lambda_1, \gamma\}_{t=1}^{T}$, as in the simpler case, above.
- This is better, but the biggest issue is that $\beta_i$ can be biased in the first stage regression.

# Extensions using the Fama and Macbeth Approach

- We can add extra explanatory variables. Why? Sharpe-Linter and Black CAPM $\Rightarrow$ market portfolio is MV efficient. $\Rightarrow$ differences in expected returns across securities and portfolios are solely explained by differences in beta $\Leftrightarrow$ other variables should add nothing to expected returns. (i.e., once we control for beta, no other characteristics of stocks should affect the required return)
- To test linearity we can add squared betas.
- To test if market beta measures the risk needed to explain expected returns, we can add residual variances from regressions on the market rate.
- Again, the general conclusion is that neither add to explanation.

1. Central CAPM prediction that market betas explain expected returns and that beta risk premium is positive, "seem to hold".
2. Specific Sharpe-Linter CAPM prediction, that premium per unit of beta is the expected market return minus $r$ is rejected.

# Size Effects and Other Challenges to CAPM

1. Basu (1977, JF) sorts stocks by price-earnings ratio ⇒ future returns on high E/P greater than the CAPM prediction.
2. Banz (1981, JPE) sorts stocks on (price) time (shares outstanding) [i.e. market capitalization] ⇒ average returns on small stocks greater than CAPM prediction.
3. Also debt-equity ratio ⇒ high debt/equity greater than market betas

# More Details on Banz

- Banz (1981, JPE) uses data from 1936-1975.
- He notes that average returns to stocks of small firms ($\Leftrightarrow$ those with low values of market equity) was substantially higher than the average return to stocks of large firms, after adjusting for risk.
- He splits the data into 5 sub-groups using historical betas; and 5 further sub-groups, based on market value of firms equity $\Rightarrow$ 25 data points.
- The idea is the following:

1. Ratios involving stock prices contain information about expected returns missed by market betas.
2. Stock prices depend not only on expected cash flow it provides, but also on expected returns that discount expected cash flows back to the present.

# Sample Period Problems

- Consider the following:

1. Fama and Macbeth (1973) - use 1926-66 data and find a positive relation between risk and return.
2. Fama and French (1992, JF) - use 1963-90 - and find no relation between risk and return.
3. But they also check 1941-1965 data and find a positive relation.

# CAPM issues

- Basic issues with CAPM (re-cap):

1. Small stocks deliver higher average returns than their betas can justify (Banz).
2. Stocks with high past betas had average returns no higher than stocks of equal size with low past betas.
3. Investors tilt their stock portfolios to small stocks.

# Good Beta vs Bad Beta Approach

- Campbell and Vuolteenhao's (2004, AER; not in the textbook) idea is to explain size and value anomalies.
- In general, small stocks have higher bad-betas and this explains their higher average returns (growth stocks have good betas).
- They use a "two beta model". The required return on a stock is not determined by overall beta but rather two possibilities.

1. A "bad" beta $\Leftrightarrow$ cash-flow beta ($\Longleftrightarrow$ news about future cash flow) - has a high price of risk.
2. A "good" beta $\Leftrightarrow$ discount rate beta ($\Leftrightarrow$ news about...) - has a relatively low price of risk.

# Good Beta, Bad Beta Approach

- Loosely, the two betas have different significance for investors and CV claim that this eliminates the tilt problem on size pointed out by Banz.
- Small stocks - higher cash flow betas; large stocks - low cash flow betas.
- The poor performance of CAPM explained by fact that growth stocks and high past beta stocks have mostly "good" betas with low risk prices.

# We Covered

- Mean-Variance Model and derivation of market Beta.
- Fama and Macbeth (1973) approach to testing CAPM.
- Size effects; Banz (1981).
- Good beta versus bad beta; Campbell and Vuolteenhao (2004).