Multi-variable Calculus and Optimization

Dudley Cooke

Trinity College Dublin

- **Reading**

1. Chapter 7.4-7.6, 8 and 11 (section 6 has lots of economic examples) of CW
2. Chapters 14, 15 and 16 of PR

- **Plan**

1. Partial differentiation and the Jacobian matrix
2. The Hessian matrix and concavity and convexity of a function
3. Optimization (profit maximization)
4. Implicit functions (indifference curves and comparative statics)

Multi-variable Calculus

- Functions where the input consists of many variables are common in economics.
- For example, a consumer's utility is a function of all the goods she consumes. So if there are n goods, then her well-being or utility is a function of the quantities $(c_1, \ldots, c_n)$ she consumes of the n goods. We represent this by writing $U = u(c_1, \ldots, c_n)$
- Another example is when a firm's production is a function of the quantities of all the inputs it uses. So, if $(x_1, \ldots, x_n)$ are the quantities of the inputs used by the firm and y is the level of output produced, then we have $y = f(x_1, \ldots, x_n)$.
- We want to extend the calculus tools studied earlier to such functions.

Partial Derivatives

- Consider a function $y = f(x_1, \ldots, x_n)$.
- The *partial derivative of f with respect to x_i* is the derivative of f with respect to x_i treating all other variables as constants and is denoted,

$$\frac{\partial f}{\partial x_i} \text{ or } f_{x_i}$$

- Consider the following function: $y = f(u, v) = (u + 4)(6u + v)$.
- We can apply the usual rules of differentiation, now with two possibilities:

$$\frac{\partial f}{\partial u} = 1(6u + v) + 6(u + 4) = 12u + v + 24$$
$$\frac{\partial f}{\partial v} = 0(6u + v) + 1(u + 4) = u + 4$$

Partial Derivatives - Economic Examples

- Again, suppose $y = f(x_1, \ldots, x_n)$. Assume $n = 2$ and $x_1 = K$ and $x_2 = L$; that is, $y = f(K, L)$.
- Then assume the following functional form: $f(K, L) = K^{0.25}L^{0.75}$; i.e., a Cobb-Douglas production function.
- The partial derivatives are given by,

$$\frac{\partial f}{\partial K} = 0.25K^{-0.75}L^{0.75}, \ \frac{\partial f}{\partial L} = 0.75K^{0.25}L^{-0.25}$$

- Alternatively; suppose utility is $U = c_a^{\alpha} c_b^{1-\alpha}$, where $a =$apples, $b =$bananas. We find:

$$\frac{\partial U}{\partial c_a} = \alpha \left(\frac{c_b}{c_a} \right)^{1-\alpha}, \ \frac{\partial U}{\partial c_b} = (1-\alpha) \left(\frac{c_a}{c_b} \right)^{\alpha}$$

- So, for a given labor input, more capital raises output. For given consumption of apples, consuming more bananas makes you happier.

Interpretation of Partial Derivatives

- Mathematically, the partial derivative of f with respect to x_i tells us the rate of change when only the variable x_i is allowed to change.
- Economically, the partial derivatives give us useful information. In general terms:

1. With a production function, the partial derivative with respect to the input, x_i, tells us the marginal productivity of that factor, or the rate at which additional output can be produced by increasing x_i, holding other factors constant.
2. With a utility function, the partial derivative with respect to good c_i tells us the rate at which the consumer's well being increases when she consumes additional amounts of c_i holding constant her consumption of other goods. I.e., the marginal utility of that good.

Derivative of a Linear Function

- As with functions of one variable, to gain some intuition, we start with functions that are linear.
- We motivated the notion of a derivative by saying that it was the slope of the line which "looked like the function around the point x_0."
- When we have n variables, the natural notion of a "line" is given by

$$y = a_0 + a_1 x_1 + \ldots + a_n x_n$$

- When there are two variables, x_1 and x_2, the function $y = a_0 + a_1 x_1 + a_2 x_2$ is the equation of a *plane*.
- In general, the function $y = a_0 + a_1 x_1 + \ldots + a_n x_n$ is referred to simply as the equation of a plane.

Two Variable Linear Function

- Consider the plane $y = a_0 + a_1 x_1 + a_2 x_2$.
- How does the function behave when we change x_1 and x_2? Clearly, if dx_1 and dx_2 are the amounts by which we change x_1 and x_2, we have,

$$dy = a_1 dx_1 + a_2 dx_2$$

- Note furthermore that the partials are,

$$\frac{\partial y}{\partial x_1} = a_1 \text{ and } \frac{\partial y}{\partial x_2} = a_2$$

- We can then write,

$$dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2$$

Jacobian Matrix

- We can take this result (i.e., that $dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2$) and recast it in matrix (here, vector) form.

$$dy = \underbrace{\left[\begin{array}{cc} \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_2} \end{array} \right]}_{\equiv \underline{J}} \left[\begin{array}{c} dx_1 \\ dx_2 \end{array} \right]$$

- We call $\underline{J}$ the Jacobian.
- Take our previous example. The vector $\left[\begin{array}{cc} \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_2} \end{array} \right] = \left[\begin{array}{cc} a_1 & a_2 \end{array} \right]$ is what we can think of as the *derivative* of the plane.
- This vector tells us the rates of change in the directions x_1 and x_2. The total change is therefore computed as $a_1 dx_1 + a_2 dx_2$.

General Functions

- Now consider a more general two variable function, $y = f(x_1, x_2)$.
- With a general function $y = f(x_1, x_2)$, the idea is to find a plane which looks locally like the function around the point (x_1, x_2).
- Since the partial derivatives give the rates of change in x_1 and x_2, it makes sense to pick the appropriate plane which passes through the point (x_1, x_2) and has slopes $\partial f / \partial x_1$ and $\partial f / \partial x_2$ in the two directions.
- The *derivative* (or the *total derivative*) of the function $f(x_1, x_2)$ at (x_1, x_2) is simply the vector $\begin{bmatrix} \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_2} \end{bmatrix}$ where the partial derivatives are evaluated at the point (x_1, x_2).
- We can interpret the derivative as the slopes in the two directions of the plane which looks "like the function" around the point (x_1, x_2).

An Example

- When we have a function $y = f(x_1, \ldots, x_n)$, the derivative at $(x_1, \ldots, x_n)$ is simply the vector $\begin{bmatrix} \frac{\partial y}{\partial x_1} & \ldots & \frac{\partial y}{\partial x_n} \end{bmatrix}$.
- Take a more concrete example. Suppose,

$$y = 8 + 4x_1^2 + 6x_2x_3 + x_3$$

- The Jacobian must be,

$$\begin{bmatrix} 8x_1 & 6x_3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_2} & \frac{\partial y}{\partial x_2} \end{bmatrix}$$

- So we can write,

$$dy = \begin{bmatrix} 8x_1 & 6 & 1 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

An Economic Example

- Consider the utility function from before; $U = c_a^{\alpha} c_b^{1-\alpha}$, where $a =$apples, $b =$bananas. The Jacobian is a vector of the partials (which we already computed).
- The change in utility is therefore given by the following.

$$
\begin{aligned}
dU &= \begin{bmatrix} \frac{\partial U}{\partial c_a} & \frac{\partial U}{\partial c_b} \end{bmatrix} \begin{bmatrix} dc_a \\ dc_b \end{bmatrix} \\
&= \begin{bmatrix} \alpha \left(\frac{c_b}{c_a} \right)^{1-\alpha} & (1-\alpha) \left(\frac{c_a}{c_b} \right)^{\alpha} \end{bmatrix} \begin{bmatrix} dc_a \\ dc_b \end{bmatrix}
\end{aligned}
$$

- The same is true for the Cobb-Douglas production function, $f(K, L) = K^{0.25} L^{0.75}$.

Second-Order Partial Derivatives

- We now move on to second-order partial derivatives. Our motivation will be economic problems/interpretation.
- Given a function $f(x_1, \ldots, x_n)$, the second-order derivative $\frac{\partial^2 f}{\partial x_i x_j}$ is the partial derivative of $\frac{\partial f}{\partial x_i}$ with respect to x_j.
- The above may suggest that the order in which the derivatives are taken matters: in other words, the partial derivative of $\frac{\partial f}{\partial x_i}$ with respect to x_j is different from the partial derivative of $\frac{\partial f}{\partial x_j}$ with respect to x_i.
- While this can happen, it turns out that if the function $f(x_1, \ldots, x_n)$ is *well-behaved* then the order of differentiation does not matter. This result is called *Young's Theorem*.
- We shall only be dealing with well-behaved functions (i.e., Young's Theorem will always hold).

Economic Example

- For the production function $y = K^{0.25}L^{0.75}$, we can evaluate the second-order partial derivative, $\frac{\partial^2 y}{\partial K \partial L}$, in two different ways.
- First, since $\frac{\partial y}{\partial K} = 0.25K^{-0.75}L^{0.75}$, taking the partial derivative of this with respect to L, we get,

$$\frac{\partial}{\partial L}\frac{\partial y}{\partial K} = \frac{3}{16}K^{-0.75}L^{-0.25}$$

- And since $\frac{\partial f}{\partial L} = 0.75K^{0.25}L^{-0.25}$, we have,

$$\frac{\partial}{\partial K}\frac{\partial y}{\partial L} = \frac{3}{16}K^{-0.75}L^{-0.25}$$

- This illustrates Young's Theorem: no matter in which order we differentiate, we get the same answer.

Economic Interpretation

- The second-order partial derivatives can be interpreted economically.
- Our example production function is $y = K^{0.25}L^{0.75}$.
- Therefore,
$$\frac{\partial^2 y}{\partial K^2} = -\frac{3}{16}K^{-1.75}L^{0.75}$$
- This is negative, so long as $K > 0$ and $L > 0$.
- This tells us that the marginal productivity of capital decreases as we add more capital.

Economic Interpretation

- As another example, consider,

$$\frac{\partial^2 y}{\partial K \partial L} = \frac{3}{16} K^{-0.75} L^{-0.25}$$

- This is positive, so longas $K > 0, L > 0$.
- This says that the marginal productivity of labour increases as you add more capital.
- It also suggests a symmetry. The above can be interpreted as saying that the marginal productivity of capital increases when you add more labor.

Concave and Convex Functions

- Our interest in these SOCs is motivated by the local versus global max/min of functions. Previously, we related this to concavity and convexity.
- In the case of one variable, we defined a function $f(x)$ as concave if $f''(x) \leq 0$ and convex if $f''(x) \geq 0$.
- Notice that in the single-variable case, the second-order total is,

$$d^2y = f''(x)(dx)^2$$

- Hence, we can (equivalently) define a function of one variable to be concave if $d^2y \leq 0$ and convex if $d^2y \geq 0$. The advantage of writing it in this way is that we can extend this definition to functions of many variables.

Second-Order Total Differential

- Suppose $y = f(x_1, x_2)$.
- The first order differential is $dy = f_{x_1} dx_1 + f_{x_2} dx_2$.
- Think of dy as a function. It's differential is,

$$d(dy) = [f_{x_1 x_1} dx_1 + f_{x_1 x_2} dx_2] \, dx_1 + [f_{x_1 x_2} dx_1 + f_{x_2 x_2} dx_2] \, dx_2$$

- Collecting terms,

$$d^2 y = f_{x_1 x_1} (dx_1)^2 + 2 f_{x_1 x_2} dx_1 dx_2 + f_{x_2 x_2} (dx_2)^2$$

- The second-order total differential depends on the second-order partial derivatives of $f(x_1, x_2)$.

Second-Order Total Differential

- **Corollary:** If we have a general function $y = f(x_1, \ldots, x_n)$, one can use a similar procedure to get the formula for the second-order total differential.
- This is a little more complicated but it can be written compactly as,

$$d^2y = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{x_i x_j} dx_i dx_j$$

- E.g., $y = f(x_1, x_2, x_3)$ the expression becomes,

$$\begin{aligned} d^2y = {} & f_{x_1 x_1}(dx_1)^2 + f_{x_2 x_2}(dx_2)^2 + f_{x_3 x_3}(dx_3)^2 \\ & + 2f_{x_1 x_2} dx_1 dx_2 + f_{x_1 x_3} dx_1 dx_3 + f_{x_2 x_3} dx_2 dx_3 \end{aligned}$$

- Again, we care about this because a function $y = f(x_1, \ldots, x_n)$ will be *concave* if $d^2y \leq 0$ and *convex* if $d^2y \geq 0$.

Maxima/Minima/Saddle Points

- As in the single variable case, we are really after maxima and minima. The first order conditions alone cannot distinguish between local maxima and local minima.
- Likewise, the first order conditions cannot identify whether a candidate solution is a local or global maxima. We thus need second order conditions to help us.
- For a point to be a local maxima, we must have $d^2y < 0$ for any vector of small changes $(dx_1, \ldots, dx_n)$; that is, y needs to be a strictly concave function.
- Similarly, for a point to be a local minima, we must have $d^2y > 0$ for any vector of small changes $(dx_1, \ldots, dx_n)$; that is, y needs to be a strictly convex function.

Hessian Matrix - Two Variable Case

- As things stand, none of this is not particularly useful since it is not clear how to go about verifying that the second-order total differential of a function of n variables is never positive or never negative.
- However, notice that we can write the second order differential of a function of two variables (that is, $d^2y = f_{11}(dx_1)^2 + 2f_{12}dx_1dx_2 + f_{22}(dx_2)^2$) in matrix form in the following way.

$$d^2y = \begin{bmatrix} dx_1 & dx_2 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$$

- The matrix $\begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix}$ is called the *Hessian matrix*, H. It is simply the matrix of second-order partial derivatives.
- Whether $d^2y > 0$ or $d^2y < 0$ leads to specific restrictions on the Hessian.

Hessian Matrix - General Case

- In the general case, the second order differential can be written as,

$$d^2y = \begin{bmatrix} dx_1 & dx_2 & \dots & dx_n \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ f_{12} & f_{22} & \dots & f_{2n} \\ \dots & \dots & \dots & \dots \\ f_{1n} & f_{2n} & \dots & f_{nn} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix}$$

- If we want to know something about d^2y all we need to do is evaluate the Hessian.

- Returning to the Cobb-Douglas utility case; $U = c_a^{\alpha} c_b^{\beta}$, where $a =$apples, $b =$bananas. The first-order differential was,

$$dU = \begin{bmatrix} \alpha c_a^{\alpha-1} c_b^{\beta} & \beta c_a^{\alpha} c_b^{\beta-1} \end{bmatrix} \begin{bmatrix} dc_a \\ dc_b \end{bmatrix}$$

- The second-order differential is,

$$d^2 U = \begin{bmatrix} dc_a & dc_b \end{bmatrix} \begin{bmatrix} \frac{\partial^2 U}{\partial c_a \partial c_a} & \frac{\partial^2 U}{\partial c_a \partial c_b} \\ \frac{\partial^2 U}{\partial c_b \partial c_a} & \frac{\partial^2 U}{\partial c_b \partial c_b} \end{bmatrix} \begin{bmatrix} dc_a \\ dc_b \end{bmatrix}$$

- where,

$$\begin{bmatrix} \frac{\partial^2 U}{\partial c_a \partial c_a} & \frac{\partial^2 U}{\partial c_a \partial c_b} \\ \frac{\partial^2 U}{\partial c_b \partial c_a} & \frac{\partial^2 U}{\partial c_b \partial c_b} \end{bmatrix} = \begin{bmatrix} \alpha(\alpha-1) c_a^{\alpha-2} c_b^{\beta} & \alpha\beta c_a^{\alpha-1} c_b^{\beta-1} \\ \alpha\beta c_a^{\alpha-1} c_b^{\beta-1} & \beta(\beta-1) c_a^{\alpha} c_b^{\beta-2} \end{bmatrix}$$

Positive Definite and Negative Definite Matrices

- The Hessian matrix is square. We can say something about square matrices:
- A symmetric, square matrix, A_n, is said to be *positive definite* if for every column vector $x \neq 0$ (dimension $1 \times n$) we have $x^T Ax > 0$.
- A symmetric, square matrix A_n is *negative definite* if for every column vector $x \neq 0$ (dimension $1 \times n$), we have $x^T Ax < 0$.
- To check for concavity/convexity we evaluate the Hessian. But the Hessian (for our purposes) can be PD or ND.
- If the Hessian is ND the function is concave. If the Hessian is PD the function is convex. All we need to be able to do then is detemine whether a given matrix is ND or PD.

Checking for PD-ness and ND-ness

- Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix}$ be a $n \times n$ symmetric square matrix.
- The *principal minors* of A are the n determinants,

$$a_{11}, \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}, \dots, |A|$$

- The matrix A is positive-definite if and only if all the principal minors are positive.
- The matrix A is negative-definite if and only if its principal minors alternate in sign, starting with a negative.

Examples of PD and ND Matrices

- One can see how this works if we look at the matrices I_n and $-I_n$. The principal minors of I are uniformly 1; on the other hand the principal minors of $-I$ start with -1 and then alternate in sign. Thus, the matrices I_n and $-I_n$ are trivially positive-definite and negative-definite.
- Consider, $\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$. One can easily show that the matrix is positive-definite. How? Well, the principle minors are 1 and 3.
- The matrix $\begin{bmatrix} -1 & 1 \\ 1 & -4 \end{bmatrix}$ is negative-definite as the principle minors are -1 and 3.
- A matrix may be neither positive-definite nor negative-definite. For instance, consider the matrix $\begin{bmatrix} -1 & 1 \\ 1 & 4 \end{bmatrix}$.

ND and PD Relation to Concavity and Convexity

- We defined a function to be concave if $d^2y \leq 0$ for all x and to be convex if $d^2y \geq 0$ for all x.
- If the function satisfies a stronger condition – $d^2y < 0$ for all x – then it will be said to be *strictly concave*. Analogously, if $d^2y > 0$ for all x, then it will be said to be *strictly convex*.
- We can also relate the notions of positive-definiteness and negative-definiteness to strict concavity and strict convexity.

1. A function f is strictly concave if the Hessian matrix is always negative definite.
2. A function f is strictly convex if the Hessian matrix is always positive definite.

Example

- Consider the following. Is $f(x,y) = x^{1/4}y^{1/2}$ concave or convex over the domain $(x,y) \geq 0$?
- Compute the second order partials. E.g., $f_{xy}(x,y) = f_{yx}(x,y) = (1/8)\,x^{-3/4}y^{-1/2}$ and $f_{xx}(x,y) = -(3/4)(1/4)\,x^{-7/4}y^{1/2}$. So,
$$H = \begin{bmatrix} -(3/4)(1/4)\,x^{-7/4}y^{1/2} & (1/8)\,x^{-3/4}y^{-1/2} \\ (1/8)\,x^{-3/4}y^{-1/2} & -(1/4)\,x^{1/4}y^{-3/2} \end{bmatrix}$$
- The principle minors are, $-(3/4)(1/4)\,x^{-7/4}y^{1/2} < 0$ and,
$$\begin{vmatrix} -(3/4)(1/4)\,x^{-7/4}y^{1/2} & (1/8)\,x^{-3/4}y^{-1/2} \\ (1/8)\,x^{-3/4}y^{-1/2} & -(1/4)\,x^{1/4}y^{-3/2} \end{vmatrix}$$
$$= (3/64)\,x^{-7/4}y^{1/2}x^{1/4}y^{-3/2} - (1/64)\,x^{-3/4}y^{-1/2}x^{-3/4}y^{-1/2}$$
$$= (1/64)\left[2x^{-6/4}\right]/y > 0$$
- We conclude $x^{1/4}y^{1/2}$ is convex over the domain $(x,y) \geq 0$.

Unconstrained Optimization with Multiple Variables

- We now link concavity and convexity to optimization problems in economics.
- Consider the general maximization problem,

$$\max f(x_1, \ldots, x_n)$$

- The first order conditions for maximization can be identified from the condition that the first order differential should be zero at the optimal point.
- That is, a vector of small changes $(dx_1, ..., dx_n)$ should not change the value of the function.
- We thus have

$$dy = f_{x_1} dx_1 + f_{x_2} dx_2 + \ldots + f_{x_n} dx_n = 0$$

First Order Conditions

- Suppose that we have a function of 2 variables $y = f(x_1, x_2)$. Then, the first order condition $dy = 0$ implies that $f_{x_1} dx_1 + f_{x_2} dx_2 = 0$.
- Therefore, the only way we can have $f_{x_1} dx_1 + f_{x_2} dx_2 = 0$ is when $f_{x_1} = 0$ and $f_{x_2} = 0$.
- By the same logic, for a general function $y = f(x_1, \ldots, x_n)$, the first order conditions are

$$f_{x_1} = 0,\ f_{x_2} = 0, \ldots,\ f_{x_n} = 0$$

- Note that these conditions are *necessary* conditions in that they hold for minimization problems also.

Relation to SOCs for Optimization

- We can now summarize the first and second order conditions for an optimization problem in the following way:
- If $(x_1^*, \ldots, x_n^*)$ is a local maximum, then the first order conditions imply that $f_i(x_1^*, \ldots, x_n^*) = 0$ for $i = 1, \ldots, n$. The second order condition implies that the Hessian matrix evaluated at $(x_1^*, \ldots, x_n^*)$ must be negative definite.
- If $(x_1^*, \ldots, x_n^*)$ is a local minimum, then the first order conditions imply that $f_i(x_1^*, \ldots, x_n^*) = 0$ for $i = 1, \ldots, n$. The second order condition implies that the Hessian matrix evaluated at $(x_1^*, \ldots, x_n^*)$ must be positive definite.

Economic Example: Profit Maximization

- To better understand all of this, we can look at a firm's optimization problem.
- Suppose a firm can sell it's output at \$10 per unit and that it's production function is given by $y = K^{1/4}L^{1/2}$.
- What combination of capital and labour should the firm use so as to maximize profits assuming that capital costs \$2 per unit and labour \$1 per unit?
- The firm's profits are given by

$$\Pi(K, L) = \underbrace{10y - 2K - L}_{\text{revenue minus costs}} = 10K^{1/4}L^{1/2} - 2K - L$$

- This is a problem of unconstrained optimization with multiple variables.

Economic Example Continued

- We can use the first order conditions to obtain **potential candidates** for optimization, as with one variable.
- For the firm we have the following profit maximization problem:

$$\Pi(K, L) = 10K^{1/4}L^{1/2} - 2K - L$$

- The first order conditions are

$$\frac{\partial \Pi}{\partial K} = (5/2)K^{-3/4}L^{1/2} - 2 = 0$$
$$\frac{\partial \Pi}{\partial L} = 5K^{1/4}L^{-1/2} - 1 = 0$$

Economic Example Continued

- Dividing the first first-order condition by the second first-order condition gives us $(1/2)L/K = 2$, or,

$$L = 4K$$

- Substituting this into the second first-order condition, we get $5K^{1/4}(4K)^{-1/2} = 1$, or $K^{-1/4} = 2/5$, or,

$$K = (5/2)^4 = 625/16$$

- Finally, we have,

$$L = 625/4$$

- This is our candidate solution for a maximum. But to verify that it really is the maximum we need to check the second-order conditions.

Economic Example Continued

- The candidate solutions are, $K^* = 625/16$ and $L^* = 625/4$.
- The Hessian matrix at any (K, L) is given by

$$H = \begin{bmatrix} -\frac{15}{8}K^{-7/4}L^{1/2} & \frac{5}{4}K^{-3/4}L^{-1/2} \\ \frac{5}{4}K^{-3/4}L^{-1/2} & -\frac{5}{2}K^{1/4}L^{-3/2} \end{bmatrix}$$

- Note that since $K > 0, L > 0$.

$$f_{11} = -\frac{15}{8}K^{-7/4}L^{1/2} < 0$$

$$\begin{aligned} |H| &= -\left(-\frac{5}{2}\right)\frac{15}{8}K^{-7/4+1/4}L^{-3/2+1/2} - \left(\frac{5}{4}\right)^2 K^{-3/4-3/4}L^{-1/2-1/2} \\ &= (75/16)K^{-3/2}L^{-1} - (25/16)K^{-3/2}L^{-1} \\ &= (50/16)K^{-3/2}L^{-1} > 0 \end{aligned}$$

- This shows that the second order conditions are satisfied at (K^*, L^*), which shows it is a local maximum.

Implicit Functions and Comparative Statics

- So far, we have consider the interpretation of functions with more than one variable through the Jacobian and the Hessian.
- We have also related the Hessian to the notions of concavity and convexity and maxima and minima, via an optimization problem.
- However, when we deal with functions of more than one variable, we also run into other complications.
- Basically, we tend to think of the LHS variable of a function as endogenous and the RHS variable(s) as exogenous. But sometimes exogenous variables and endogenous variables cannot be separated and we need to differentiate implicitly.
- Implicit differentiation is also useful when thinking about utility functions (indifference curves) and production functions (isoquant curves).

Implicit Function Theorem

- An example of an *implicit function*, i.e., one where the exogenous and endogenous variables cannot be separated, is,

$$F(x, y) = x^2 + xye^y + y^2x - 10 = 0$$

- It is not clear how to separate out y from x or even whether it can be done. However, we may still want to find out how the endogenous variable changes when the exogenous variable changes.
- Under certain circumstances, it turns out that even when we cannot separate out the variables, we can nonetheless find the derivative dy/dx .

Implicit Function Theorem

- The following result enables us to find the derivatives for implicit functions.
- **Theorem**: Let $f(x, y) = 0$ be an implicit function which is *continuously differentiable* and (x_0, y_0) be such that $f(x_0, y_0) = 0$. Suppose that $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$. Then,

$$\frac{dy}{dx}\bigg|_{(x_0,y_0)} = -\frac{\frac{\partial f}{\partial x}(x_0, y_0)}{\frac{\partial f}{\partial y}(x_0, y_0)}$$

- So, without specifying x and y we can find $\frac{dy}{dx}$ at a given point, here (x_0, y_0).

Abstract Example of Implicit Function Theorem

- Consider $f(x,y) = x^2 + xye^y + y^2x - 10 = 0$. Suppose we want to evaluate the derivative at some point (x_0, y_0) such that $f(x_0, y_0) = 0$.
- Using the rules of differentiation,

$$\frac{\partial f}{\partial y} = xe^y + xye^y + 2xy$$

- This *can* be zero for some x and y but assume that the point (x_0, y_0) is not one of them. Now differentiate with respect to x,

$$\frac{\partial f}{\partial x} = 2x + ye^y + y^2$$

- The implicit function theorem allows us to write the following.

$$\left.\frac{dy}{dx}\right|_{(x_0,y_0)} = -\frac{2x_0 + y_0 e^{y_0} + y_0^2}{x_0 e^{y_0} + x_0 y_0 e^{y_0} + 2x_0 y_0}$$

Economic Example of Implicit Function Theorem

- One use of the implicit function theorem in Economics is to find the slopes of indifference curves and isoquants.
- Consider the production function used earlier.

$$y = K^{0.25}L^{0.75}$$

- If we fix $y = y_0$, then the combination of all (K, L) which gives y_0 units of output is called an isoquant.
- Suppose (K_0, L_0) is one such combination.
- We want to find the slope of the isoquant, that is $\frac{dK}{dL}\Big|_{(K_0,L_0)}$.

Economic Example Continued

- Setting $y = y_0$, we can write the production function in implicit function form as,

$$F(K, L) = K^{0.25} L^{0.75} - y_0 = 0$$

- Using the values for the partial derivatives found earlier, we have,

$$\frac{dK}{dL}\bigg|_{(K_0, L_0)} = -\frac{0.75 K_0^{0.25} L_0^{-0.25}}{0.25 K_0^{-0.75} L_0^{0.75}} = -3\frac{K_0}{L_0}$$

- Note that $K_0 \neq 0$, $L_0 \neq 0$.
- This equation tells us the slope of the isoquant at (K_0, L_0).

Indifference Curves

- As a final example, consider the following utility function: $U = \left[\alpha c_a^\eta + \beta c_b^\eta\right]^{1/\eta}$, where $a =$apples, $b =$bananas. This is called a constant elasticity of substitution utility function. When $\eta = 0$ we get back to the Cobb-Douglas utility function.
- Holding utility fixed, we have the following: $\left[\alpha c_a^\eta + \beta c_b^\eta\right]^{1/\eta} - U_0 = 0$. Now use the implicit function theorem.

$$\left.\frac{dc_a}{dc_b}\right|_{(c_{a,0},c_{b,0})} = -\frac{\frac{\partial U}{\partial c_a}(c_{a,0}, c_{b,0})}{\frac{\partial U}{\partial c_b}(c_{a,0}, c_{b,0})}$$

$$= -\frac{\alpha}{\beta}\left(\frac{c_a}{c_b}\right)^{\eta-1}$$

- Suppose we look at the MRS at the point (tc_a, tc_b). The indifference curves have the same slope at (c_a, c_b) and at (tc_a, tc_b) for any $t > 0$. The CES utility function is an example of preferences that are homothetic.

Comparative Statics and Implicit Relations: Supply and Demand

- One question we also often need to answer is how underlying parameters affect the equilibrium of a supply and demand system.
- Suppose that we have a market where the demand and supply functions are given by the following.

$$D = a - bp + cy,\ S = \alpha + \beta p$$

- The parameters a, b, c, α, β are all positive constants.
- The equilibrium price is,

$$p^* = \frac{a - \alpha + cy}{b + \beta}$$

- We can now check how the underlying parameters $(a, b, c, y, \alpha, \beta)$ affect the equilibrium price.

Comparative Statics: Supply and Demand

- In our simple example, we can get the partial derivative,

$$\partial p^* / \partial a = 1/(b + \beta)$$

- All other things remaining constant, an increase in a, increases the equilibrium price.
- We can similarly find the impact of changes in other parameters on the equilibrium price.
- Importantly, we can also, in many cases, conduct the comparative statics without solving for the equilibrium values of the endogenous variables.

General Function Example of Supply and Demand

- Consider the following system more general demand and supply system.

$$q_d = D(p, T), q_s = S(p, T)$$

- Here, T is the temperature on a given day. We assume that $D_p < 0$, $D_T > 0$, $S_p > 0$, $S_T < 0$.
- Equilibrium requires that $D(p, T) = S(p, T)$ but we cannot compute the equilibrium price explicitly.
- Denote the equilibrium price $p^*(T)$: observe that it is a function of the parameter T.
- Inserting $p^*(T)$ into the equilibrium condition, we have,

$$D(p^*(T), T) = S(p^*(T), T)$$

General Function Example of Supply and Demand

- We differentiate both sides with respect to T which gives,

$$D_p \frac{dp^*}{dT} + D_T = S_p \frac{dp^*}{dT} + S_T$$

- Rearranging, we get

$$\frac{dp^*}{dT} = \frac{S_T - D_T}{D_p - S_p}$$

- Note that since $S_T < 0, D_T > 0, D_p < 0, S_p > 0$ it follows that $dp^*/dT > 0$.
- Hence, even without being able to compute $p^*(T)$ explicitly, we can still say that the equilibrium price increases following an increase in the temperature.

Example Continued

- Can we say anything about the equilibrium quantity?
- Note that the equilibrium quantity must satisfy,

$$q^*(T) = D(p^*(T), T)$$

- Differentiating with respect to T we get

$$\frac{dq^*}{dT} = D_p \frac{dp^*}{dT} + D_T = \frac{D_p S_T - S_p D_T}{D_p - S_p}$$

- In this case, the denominator is positive but we cannot say anything about the numerator unless we have more specific information about the magnitudes of D_p, S_p, D_T and S_T.

General Formulation

- Suppose we have a general set of equilibrium conditions given by

$$
\begin{aligned}
f^1(x_1, \ldots, x_n; \alpha_1, \ldots, \alpha_m) &= 0 \\
f^2(x_1, \ldots, x_n; \alpha_1, \ldots, \alpha_m) &= 0 \\
\ldots &= \ldots \\
f^n(x_1, \ldots, x_n; \alpha_1, \ldots, \alpha_m) &= 0
\end{aligned}
$$

- Let us suppose that we can solve this set of equations to get $(x_1^*, \ldots, x_n^*)$. Note that each x_i^* will be a function of all the underlying parameters $(\alpha_1, \ldots, \alpha_n)$.
- In comparative statics, we typically are interested in knowing how $(x_1^*, \ldots, x_n^*)$ change when there is a small change in one of the parameters, say α_i.

- We proceed by differentiating each of the equilibrium conditions with respect to α_i. This gives

$$
\begin{aligned}
f_1^1 \frac{dx_1^*}{d\alpha_i} + f_2^1 \frac{dx_2^*}{d\alpha_i} + \ldots + f_n^1 \frac{dx_n^*}{d\alpha_i} &= -f_{\alpha_i}^1 \\
f_1^2 \frac{dx_1^*}{d\alpha_i} + f_2^2 \frac{dx_2^*}{d\alpha_i} + \ldots + f_n^2 \frac{dx_n^*}{d\alpha_i} &= -f_{\alpha_i}^2 \\
\vdots &= \vdots \\
f_1^n \frac{dx_1^*}{d\alpha_i} + f_2^n \frac{dx_2^*}{d\alpha_i} + \ldots + f_n^n \frac{dx_n^*}{d\alpha_i} &= -f_{\alpha_i}^n
\end{aligned}
$$

General Formulation

- In matrix notation this can be written as

$$\begin{bmatrix} f_1^1 & f_2^1 & \dots & f_n^1 \\ f_1^2 & f_2^2 & \dots & f_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ f_1^n & f_2^n & \dots & f_n^n \end{bmatrix} \begin{bmatrix} \frac{dx_1^*}{d\alpha_i} \\ \frac{dx_2^*}{d\alpha_i} \\ \vdots \frac{dx_n^*}{d\alpha_i} \end{bmatrix} = \begin{bmatrix} -f_{\alpha_i}^1 \\ -f_{\alpha_i}^2 \\ \vdots \\ -f_{\alpha_i}^n \end{bmatrix}$$

- How do we solve for $dx_j^* / d\alpha_i$? Using the usual methods (like Cramer's rule).
- All of this gives us a very powerful tool more analyzing relatively complicated economics models in a simple way.

Roundup

- You should now be able to do/understand the following:

1. Partial differentiation and the Jacobian matrix
2. The Hessian matrix and concavity and convexity of a function
3. Optimization (profit maximization)
4. Implicit functions (indifference curves and comparative statics)