Testing for Long Memory and Nonlinear Time Series: A Demand for Money Study

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Abstract

This paper draws attention to the limitations of the standard unit root/cointegration approach to economic and financial modelling, and to some of the alternatives based on the idea of fractional integration, long memory models, and the random field regression approach to nonlinearity. Following brief explanations of fractional integration and random field regression, and the methods of applying them, selected techniques are applied to a demand for money dataset. Comparisons of the results from this illustrative case study are presented, and conclusions are drawn that should aid practitioners in applied time-series econometrics.

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by

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1 Introduction

The importance of the concepts of stationarity and regime stability in economic and financial time-series modelling is well established. However, recent concerns about the interrelationship between these two concepts, and the associated problems for applied work, have ensured that they remain a significant focus for research. Early studies, such as those by Bhattacharya, et al. (1983) and Perron (1989), highlighted the difficulty of distinguishing between time series generated by difference stationary processes and those generated by nonlinear but stationary processes. Since then, an increasing research emphasis has been on the problem of distinguishing between long memory and nonlinearity. The developing interest in long memory models has been stimulated, in particular, by a growing awareness of the limitations of the simple $I(1)/I(0)$ framework. For example, Baillie and Bollerslev (2000) and Maynard and Phillips (2001) show how the low power of familiar unit root tests, such as those introduced by Dickey and Fuller (1981), could lead to incorrect inference in the Fama (1984) regression model of the relationship between future spot and forward exchange rates, and how the empirical work could be set in a framework of fractional integration using a long memory model. Long memory models and fractional (co)integration are now popular in several other areas of the applied literature; see, for example, Gil-Alana (2003), Liu and Chou (2003), Dittmann (2004), and Masih and Masih (2004). A major problem with such models is that it is not easy to distinguish them empirically from models with regime switching or more general nonlinearities; see, for instance, Diebold and Inoue (2001).

In the theoretical literature, two main strands of discussion have developed. The first is that of testing for difference stationarity when the processes are in fact nonstationary; see Perron and Qu (2004) for references. The second concerns testing for structural breaks when long memory is a possibility; see Nunes, et al. (1995), Krämmer and Sibbertsen (2002), and Hsu (2001). Recent work by Mayoral (2005) and Dolado, et al. (2005b) has developed specific tests for difference stationarity against the alternative of stationarity with a structural break. All of these studies use conventional parametric techniques for either modelling or testing for nonlinearities. The recent development of random field regression has also provided a suite of tests for

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structural breaks, nonlinearity and time-varying parameters; see, for example, Hamilton (2001) and Dahl (2002). The strength of this alternative approach is that it does not rely on any functional form being specified prior to estimation.

The purpose of this paper is to compare the performance of traditional integration analysis, the fractional integration approach and random field regression-based inference, using a standard economic model and a well-known time-series dataset. The discussion is structured as follows. In Section 2, the theoretical background to fractional integration and random field regression is briefly explained. In Section 3, the three techniques are applied to the Johansen and Juselius (1990) money demand data. Finally, in Section 4, the results of the analysis are discussed and some practical conclusions drawn.

2 Theoretical Background

2.1 Fractional integration

Traditional testing for a unit root means a choice between what Maynard and Phillips (2001) call 'extreme alternatives'. The standard null hypothesis is that the series under consideration has a unit root, hence is only stationary after differencing. This knife-edge restriction, as Jensen (1999) puts it, appears to be far too stringent in many cases. The long-run behaviour of the random variable y_t in the simple $AR(1)$ model with drift,

$$y_t = \alpha + \rho y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim (0, \sigma^2), \tag{1}$$

is quite different when $\rho = 1$ from when $\rho = 0.999$. In the first case the impact of the innovation ε_t is permanent, whereas in the second case the effect disappears geometrically.

To address this issue, the concept of fractional integration, introduced into time-series analysis by Granger and Joyeux (1980), has come to the fore; see the review article by Baillie (1996) for a good introduction. Put simply, in classical integration theory, a random series $\{y_t\}_{t=0}^{\infty}$ is said to be integrated of order d, where d is an integer, if the series has to be differenced d times to induce stationarity. In the case of fractional integration the restriction that d is an integer is relaxed. Applying a Taylor series expansion to $\Delta^d = (1-L)^d$ around $L = 0$, where L is the lag operator, leads to the more general differencing formula for an integrated series of order d:

$$\begin{aligned} \Delta^d y_t &= y_t - d\ y_{t-1} + \frac{1}{2!}d(d-1)y_{t-2} - \frac{1}{3!}d(d-1)(d-2)y_{t-3} + \dots \\ &\quad + \frac{(-1)^j}{j!}d(d-1)...(d-j+1)y_{t-j} + ... \end{aligned} \tag{2}$$

In the case of $0 < d < 1$, it follows that not only the immediate past value of y_t, but values from previous time periods, influence the current value. Such series are said to have long memory. If $0 < d < 0.5$, then $\{y_t\}$ is stationary; and if $0.5 \leq d < 1.0$, the series is nonstationary.

A fundamental estimation problem is posed by the fact that Equation (2) is highly nonlinear in d. Parametric approaches to the estimation of d are computationally intensive as they often involve the estimation of a $T \times T$ covariance matrix, where T is the size of the sample, and so face issues of robustness in large samples. In the case of the maximum likelihood approach, estimation also requires the stationarity restriction $0 < d < 0.5$. Nonparametric approaches have been suggested, utilizing the frequency domain. These approaches are usually robust to nonstationarity but suffer from small sample bias.

A few econometric packages provide software to handle the estimation of the fractional integration parameter, d. Initially, the software tended to be for nonparametric methods, such as the log-periodogram regression method (GPH) introduced by Geweke and Porter-Hudak (1983). Now, a wider range of methods is available. For example, in the OX-based *ARFIMA* package of Doornik and Ooms (1999), both parametric and nonparametric methods are provided. Exact maximum likelihood estimation (EML) is implemented using the approach suggested by Sowell (1992), which employs recursive evaluations of hypergeometric functions relating to the autocorrelation function (ACF) that have negligible computational cost compared to estimating the full covariance matrix for the likelihood function. To take account of the problems with nuisance parameters, as discussed by Barndorff-Nielsen and Cox (1994, Chapter 4), the An and Bloomfield (1993) modified profile likelihood estimator (MPL) is also implemented. Both the EML and MPL methods are only applicable when $d < 0.5$, so the package also provides an approximate maximum likelihood estimator based on minimizing the sum of squared naïve residuals, which was developed by Beran (1995). Chung and Baillie (1993) called this estimator the conditional sum of squares estimator but Doornik and Ooms (1999) refer to it as nonlinear least squares (NLS). To complement these parametric estimators, the *ARFIMA* package provides two standard nonparametric methods. The first is GPH and the second is the Gaussian semiparametric method (GSP) discussed in Robinson and Henry (1998). Other methods that are gaining popularity include the modified log-periodogram estimator (MLP) of Kim and Phillips (1999) and the generalized minimum distance estimator (GMD) of Mayoral (2003).

Inference is also problematical as none of the usual procedures is appropriate. The classical asymptotics of the $I(0)$ case do not apply when time series are fractionally integrated, and neither does the standard cointegration approach. In the $I(1)$ case, conventional tests depend on the statistics converging to known functionals of Brownian motion. When $d \neq 1$, however, these are replaced by functionals of fractional Brownian motion. Taking the approach of testing for $I(1)$ against $I(d)$ is also problematical, since tests such as the ADF test of Dickey and Fuller (1981), while consistent, have very low power; see Diebold and Rudebusch (1991), and Hassler and Wolters (1994). Furthermore, the precision with which the parameters are estimated hinges on the correct specification of the model; see Hauser, et al. (1999). The situation becomes even more complex when the concept of fractional cointegration is entertained. As Phillips (2003, p. c30) points out, 'The problems presented by these models of fractional cointegration seem considerably more complex than the $I(1)/I(0)$ ... case that is now common in applications'.

An interesting group of tests, based on the ADF test, has been introduced by Dolado, et al. (2002) and these have been further developed in Dolado, et al. (2005a, 2005b) and Mayoral (2005). The latter two papers consider the important case of testing for long memory against structural breaks and use a modified ADF framework. This framework considers the t-test statistic on the ordinary least squares (OLS) estimator of ϕ in the generalized ADF equation

$$\Delta^{d_0} y_t = \phi \Delta^{d_1} y_{t-1} + \sum_{i=1}^{p} \zeta_i \Delta y_{t-i} + \varepsilon_t. \tag{3}$$

For testing purposes d_0 is set equal to 1. Dolado, et al. (2002) show that if $0.5 \leq d_1 < 1$, then the t-statistic for the null hypothesis $H_0 : \phi = 0$ has an asymptotic standard normal distribution; and if $0 \leq d_1 < 0.5$, the t-statistic follows a nonstandard distribution of fractional Brownian motion. In practice, d_1 is unknown so a consistent estimator of it has to be used. Dolado, et al. (2002) prove that provided a $T^{-\frac{1}{2}}$ consistent estimator of d_1 is used, the t-statistic has a normal distribution asymptotically for $0 \leq d_1 < 1$.

Several papers have pointed out the difficulty of distinguishing difference stationary series from nonlinear but stationary series. Examples include Perron (1989), Harrison and Bond (1992), Teverosky and Taqqu (1997), Diebold and Inoue (2001), and Perron and Qu (2004). Others consider the role of testing for difference stationary processes when the series is in fact nonlinear and stationary. Most of this research uses the alternative of a structural break in the series. In Nunes, et al. (1995), Hsu (2001), and Krämmer and Sibbertsen (2002), the reverse of testing for structural breaks in long memory models is considered. Recent work, such as that by Mayoral (2005), and Dolado, et al. (2005b), has explicitly tested for difference stationarity against the alternative of stationarity with a structural break. All of these works use traditional parametric techniques for either modelling or testing for nonlinearities. An alternative approach to the analysis of nonlinearity is to use random field regression models. The strength of this approach is that it does not rely on any functional form being specified prior to estimation and testing. Random field regression is discussed in the next subsection.

2.2 Random field regression

The paper by Hamilton (2001) introduced the idea of using random field models to estimate nonlinear economic relationships. A by-product of Hamilton's approach was a new test for nonlinearity based on the Lagrange multiplier principle.

2.2.1 Estimation

The basic random field regression model is of the form

$$y_t = \mu(\mathbf{x}_t) + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2), \quad t = 1, 2, \ldots, T, \tag{4}$$

where $\mathbf{x}_t$ is a k-vector of observations on the explanatory variables at time t, and the functional form $\mu(\mathbf{x}_t)$ is unknown, being assumed to depend on the outcome of a Gaussian random field. Stationary random fields have been used for a long time in spatial data analysis and are closely related to thin-plate splines and universal kriging. Hamilton's specification could be viewed as a thin-plate spline smoother. In his paper, Hamilton suggests representing the conditional mean function, $\mu(\mathbf{x}_t)$, as consisting of two components. The first is the usual linear form, while the second is a nonlinear component, treated as stochastic and hence unobservable. Both the linear and nonlinear components contain unknown parameters that need to be estimated. Hamilton chooses the conditional mean function to be

$$\begin{aligned} \mu(\mathbf{x}_t) &= \alpha_0 + \boldsymbol{\alpha}'\mathbf{x}_t + \lambda m(\bar{\mathbf{x}}_t), & (5)\\ \bar{\mathbf{x}}_t &= \mathbf{g} \odot \mathbf{x}_t, & (6) \end{aligned}$$

where $\mathbf{g}$ is a k-vector of parameters and $\odot$ denotes the Hadamard product. It is the function $m(\bar{\mathbf{x}}_t)$ that is specifically referred to as the random field, and there are several possible specifications for this. Hamilton (2001) showed how, under fairly general misspecification, it is possible to obtain a consistent estimator of the conditional mean under fairly weak conditions. In addition, Dahl (2002), Dahl and González-Rivera (2003) and Dahl and Hylleberg (2004) show that the random field approach has relatively better small sample fitting and forecasting abilities than a wide range of parametric and nonparametric alternatives.

Perhaps the most parsimonious representation of $m(\bar{\mathbf{x}}_t)$ is that which views it as a realization of a simple Gaussian random field. These fields have the advantage that they can be fully described by their first two moments:

$$\begin{aligned} E\left(m\left(\bar{\mathbf{x}}_t\right)\right) &= 0, & (7)\\ E\left(m\left(\bar{\mathbf{x}}_t\right) m\left(\bar{\mathbf{x}}_s\right)\right) &= H\left(d_{L^*}\left(\bar{\mathbf{x}}_t, \bar{\mathbf{x}}_s\right)\right), & (8) \end{aligned}$$

where $d_{L^*}\left(\bar{\mathbf{x}}_t, \bar{\mathbf{x}}_s\right) \in \mathbb{R}_+$ is a distance measure. An additional simplifying assumption is that the realization of the functional form occurs prior to, and therefore independently of, all observations on $\mathbf{x}_t$ and ε_t. Hamilton (2001) chooses a generalized version of the so-called spherical covariance function used in geostatistical literature:

$$H_k\left(h_{ts}\right) = \begin{cases} \frac{G_{k-1}(h_{ts},1)}{G_{k-1}(0,1)} & h_{ts} \leq 1 \\ 0 & h_{ts} > 1 \end{cases}, \tag{9}$$

$$G_k\left(h_{ts}, r\right) = \int_{h_{ts}}^{r} \left(r^2 - z^2\right)^{\frac{k}{2}} dz, \tag{10}$$

$$h_{ts} = d_{L^*}\left(\bar{\mathbf{x}}_t, \bar{\mathbf{x}}_s\right) \quad t, s = 1, 2, \ldots, T, \tag{11}$$

where $H_k(h_{ts})$ is the t-s^{th} entry in the $T \times T$ covariance matrix $\mathbf{H}$.

As $m\left(\bar{\mathbf{x}}_t\right)$ is not observable, the approach is to draw likelihood-based inference about the unknown parameters of the model, say, $\boldsymbol{\varphi} = \{\alpha_0, \boldsymbol{\alpha}, \lambda, \mathbf{g}, \sigma\}$, by observing the realizations of y_t and $\mathbf{x}_t$. The likelihood function can be derived by re-writing equations (4) and (5) for all observations, in an obvious notation, as

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\nu}, \tag{12}$$

$$\boldsymbol{\nu} \sim N\left(\mathbf{0}, \lambda^2\mathbf{H} + \sigma^2\mathbf{I}_T\right), \tag{13}$$

where

$$\boldsymbol{\nu}' = [\lambda m\left(\bar{\mathbf{x}}_1\right) + \varepsilon_1, \lambda m\left(\bar{\mathbf{x}}_2\right) + \varepsilon_2, \lambda m\left(\bar{\mathbf{x}}_3\right) + \varepsilon_3, \ldots, \lambda m\left(\bar{\mathbf{x}}_T\right) + \varepsilon_T]. \tag{14}$$

Thus maximizing the likelihood function to obtain estimates of $\boldsymbol{\varphi}$ is a generalized least squares problem. Letting $\zeta = \frac{\lambda}{\sigma}$ and $\mathbf{W}\left(\mathbf{X}; \mathbf{g}; \zeta\right) = \zeta^2\mathbf{H} + \sigma^2\mathbf{I}_T$, the profile log-likelihood function associated with the least squares problem can be obtained as

$$\eta\left(\mathbf{y}, \mathbf{X}, \mathbf{g}; \zeta\right) = -\frac{T}{2}\ln(2\pi) - \frac{T}{2}\ln\sigma^2\left(\mathbf{g}; \zeta\right) - \frac{1}{2}\ln\left|\mathbf{W}\left(\mathbf{X}; \mathbf{g}; \zeta\right)\right| - \frac{T}{2}, \tag{15}$$

while

$$\widetilde{\boldsymbol{\beta}}\left(\mathbf{g}; \zeta\right) = \left[\mathbf{X}'\mathbf{W}\left(\mathbf{X}; \mathbf{g}; \zeta\right)^{-1}\mathbf{X}\right]^{-1}\left[\mathbf{X}'\mathbf{W}\left(\mathbf{X}; \mathbf{g}; \zeta\right)^{-1}\mathbf{y}\right], \tag{16}$$

$$\widetilde{\sigma}^2\left(\mathbf{g}; \zeta\right) = \frac{1}{T}\left[\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\left(\mathbf{g}; \zeta\right)\right]'\mathbf{W}\left(\mathbf{X}; \mathbf{g}; \zeta\right)^{-1}\left[\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\left(\mathbf{g}; \zeta\right)\right]. \tag{17}$$

The profile log-likelihood function is maximized with respect to $(\mathbf{g}; \zeta)$ using standard maximization algorithms, though as Bond, et al. (2005) point out, care needs to be taken when maximizing the log-likelihood due to computational pitfalls. Once estimates for $\mathbf{g}$ and ζ have been obtained, equations (16) and (17) can be used to obtain estimates of $\boldsymbol{\beta}$ and σ.

2.2.2 Testing

The model proposed by Hamilton suggests that a simple approach to checking for nonlinearity is to test the null hypothesis $H_0 : \lambda^2 = 0$ (or $\lambda = 0$), using the Lagrange multiplier principle. Hamilton (2001) derived the appropriate score vector of first derivatives and the associated information matrix. Details of the procedure are given by Hamilton (2001), and summarized in Bond, et al. (2005), but the main steps of the test are presented here for convenience.

- Set $g_i = \frac{2}{\sqrt{ks_i^2}}$, where s_i^2 is the variance of explanatory variable x_i, excluding the constant term whose variance is zero.
- Calculate the $T \times T$ matrix, $\mathbf{H}$, whose typical element is $H_k\left(\frac{1}{2}\left\|\bar{\mathbf{x}}_t - \bar{\mathbf{x}}_s\right\|\right)$, i.e., the function $H_k(h_{ts})$ defined in (9).
- Use OLS to estimate the standard linear regression $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ and obtain the usual residuals, $\widehat{\boldsymbol{\epsilon}}$, and standard error of estimate, $\widehat{\sigma}^2 = (T - k - 1)^{-1}\widehat{\boldsymbol{\epsilon}}\,'\widehat{\boldsymbol{\epsilon}}$.

- Finally, compute the statistic

$$\varkappa^2 = \frac{\left[\widehat{\epsilon}'\mathbf{H}\widehat{\epsilon} - \widehat{\sigma}^2 tr(\mathbf{MH})\right]^2}{\widehat{\sigma}^4 \left\{2tr\left(\left[\mathbf{MHM} - (T-k-1)^{-1}\mathbf{M}tr(\mathbf{MH})\right]^2\right)\right\}}, \tag{18}$$

where $\mathbf{M} = \mathbf{I}_T - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ is the familiar symmetric idempotent matrix.

As $\varkappa^2 \overset{A}{\sim} \chi_1^2$ under the null hypothesis, linearity ($\lambda^2 = 0$) would be rejected if $\varkappa^2$ exceeded the critical value, $\chi_{1,\alpha}^2$, for the chosen level of significance, α. Otherwise the null of linearity would not be rejected. For example, at the 5 per cent significance level, the null would be rejected if $\varkappa^2 > 3.84$. In this case the alternative nonlinear specification given by (5) would be preferred. The identification of a specific form of nonlinearity is aided by the estimate of the conditional expectation $\mu(\mathbf{x}_t)$ and, specifically, the $\widetilde{\zeta}$ and $\widetilde{g}_i$. The matter is explained in Hamilton (2001, Section 5) and illustrated in the three examples in his Section 7.

3 An Empirical Case Study

3.1 Methodology

To investigate the application of both the new long memory tests and the random field approach, a standard applied economics problem, namely, the estimation of a demand for money function, is considered in this section. The well-known datasets for Denmark and Finland, provided by Johansen and Juselius (1990), are used.

The standard $I(1)/I(0)$ analysis is conducted first, in Subsection 3.2.1. The univariate analysis of the series, using the augmented Dickey-Fuller (ADF) testing strategy proposed by Dolado, et al. (1990) is implemented to determine whether the individual series are trend stationary or difference stationary.[1] The unit root tests due to Kwiatkowski, et al. (1992) (KPSS), Elliott, et al. (1996) (ERS), and Ng and Perron (2001) (NP) are also employed. Both the Engle and Granger (1987) error correction (ECM) and Johansen (1988) vector autoregression (VAR) approaches are used to investigate the possibility of cointegration, with the augmented Engle-Granger (AEG) test, the cointegrating regression Durbin-Watson (CRDW) test of Sargan and Bhargava (1983), and the ECM test due to Banerjee, et al. (1986) being used in the former case. The p-values from MacKinnon (1996), MacKinnon, et al. (1999), Ericsson and MacKinnon (2002), and standard normal tables are used, as appropriate. The effect of applying Johansen's (2002) small sample correction is also examined. This correction is based on the Bartlett (1937) correction and assumes that the errors are normal, independent and identically distributed.

[1]Although the data are quarterly, the issue of possible seasonal integration has been ignored for simplicity. A more detailed examination of the issue of unit roots might allow for this by using the procedures of Hylleberg, et al. (1990) or Osborn, et al. (1988).

Having conducted the standard cointegration analysis, the long memory and fractional integration analysis is undertaken, in Subsection 3.2.2. Only univariate analysis is attempted, due to the complexity of fractional cointegration models, which was pointed out in the quotation from Phillips (2003) given in Subsection 2.1. In particular, it seems unlikely that the series in either of the two cases considered all have the same level of fractional integration. The 'over differenced' $ARFIMA$ model, using Δy_t rather than y_t, is estimated, as recommended by Smith, et al. (1997), to avoid the problems associated with drift. Four estimates of d are calculated using the Doornik and Ooms (1999) $ARFIMA$ package, namely, the EML, NLS, GPH and GSP estimates. The fact that the first of these requires $d < 0.5$ is another reason for using the 'over-differenced' model. The MPL estimate is not obtained as there are no 'nuisance' parameters in the model being estimated. The estimates of d are then used in the fractional Dickey-Fuller (FDF) and fractional augmented Dickey-Fuller (FADF) tests, with the Schwarz (Bayesian) information criterion (SIC) being used as the basis for the choice of the lag length for the test. Finally, in Subsection 3.2.3, the random field regression approach is applied to the two cases, using the Gauss code provided by Hamilton at his website, http://weber.ucsd.edu/~jhamilto/.

3.2 Demand for money in Denmark and Finland

3.2.1 Standard analysis

The well documented instability of the demand for money function in many countries has led to several studies that place the analysis of money demand in the $I(1)/I(0)$ framework; see, for example, Astley and Haldane (1997), Fiess and McDonald (2001), Mark and Sul (2003), and Choi and Saikkonen (2004). Following Johansen and Juselius (1990), a simple demand for money function can be specified for Denmark and Finland as

$$m_t = \alpha + \beta_1 y_t + \beta_2 p_t + \beta_3 i_t + \beta_4 b_t + \varepsilon_t, \tag{19}$$

where m_t is the logarithm of some measure of money demand, y_t is the logarithm of real income, p_t is the logarithm of the inflation rate, i_t is the deposit interest rate and b_t is the bond rate at time t. For Finland, β_4 is assumed to be zero as no data are available.

Tables 1 and 2 give the results of the Dolado, et al. (1990) unit root testing strategy for the Danish and Finnish variables, respectively.[2] For Denmark, all of the data series appear to be clearly $I(1)$. However, for the Finnish data, only the m_t and y_t variables seem to be $I(1)$, though the inference is marginal for y_t. In the case of Finland's m_t variable, the constant in the ADF test is only marginally insignificant, but if it is treated as significant, the ADF test still supports the null of a unit root, with a test statistic of -0.760 and an associated p-value of 0.826. By contrast, the unit root null is rejected decisively for Finland's p_t and i_t series. It is noteworthy, though, that if, for these last two variables, the Akaike information criterion (AIC)

[2]All tables are presented in Appendix A1.

is used instead of the SIC, the choice of lag lengths for the ADF tests, and the test results, are different: the suggestion then is that, like m_t and y_t, the Finnish price and interest rate variables are also $I(1)$.

To investigate further, the KPSS, ERS and NP alternative unit root tests were conducted. While the latter two tests have as their null hypothesis that the series has a unit root, the first has the null that the series is stationary and the alternative hypothesis that it has a unit root. For the Danish data the additional tests broadly confirm the findings of Table 1. In only a few cases does the KPSS test fail to reject the null hypothesis of stationarity. One case is that of the money demand variable, m_t, when Parzen kernel estimation is used and no trend is specified. The other is that of the income variable, y_t , when a trend is allowed for in the specification. In this latter case, the result holds for any of the spectral estimation methods, but not for the moment estimators. For the Finnish data the results are less clear. For all variables, the NP test, which it has been argued has better power than standard $I(1)/I(0)$ tests, tends to reject the null hypothesis of a unit root. This is often supported by the KPSS and ERS tests.[3]

On the assumption that the variables are $I(1)$, which seems to be a far safer assumption to make for Denmark than for Finland, the Engle-Granger two-step approach to cointegration gives the estimated levels models, and associated AEG and CRDW test results for the OLS residuals, presented in Table 3. Using the 5 per cent significance level, there is little evidence for both countries that a cointegrated money demand relationship might exist. Only in the case of Finland, when p_t and i_t are ignored in view of the fact that they seem to be $I(0)$ using the Dolado, et al (1990) procedure and the supplementary unit root checks, is cointegration of m_t and y_t suggested by the AEG and CRDW tests, but even then only marginally.

The estimates of parsimonious error correction models, using the lag of the residuals from the levels regression models as the error correction terms, are given in Table 4. The models are statistically acceptable in the sense that they are supported by a range of misspecification diagnostics. Only in the case of the equations for Finland is there are marginal suggestion of heteroscedasticity. However, with R^2 values around 0.5, the fits are quite poor and there is a high incidence of insignificance of the estimated coefficients. In particular, the coefficients on the error correction terms are highly insignificant, with three out of the four being perversely signed; and the ECM test decisively rejects cointegration in all cases. Even in the one case for Finland in which the AEG and CRDW tests suggest the possibility of cointegration, the ECM test rejection is unambiguous.

The Danish data have been used extensively by Johansen and it is clear from his various results that the argument that there is a cointegrating money demand relationship depends largely on the VAR specification and the test statistic used; see Johansen (1988), Johansen and Juselius (1990), and Johansen (2002). Table 5 gives a summary of the results that can be obtained for Denmark using Johansen's approach and a VAR lag length of one, as sug-

[3]The details of the supplementary unit root tests are available on request from the authors.

gested by the SIC and the adjusted likelihood-ratio test.[4] As can be seen, a range of specifications concerning intercepts and trends was examined for variants of the model with and without seasonal dummy variables. Examination of the various VAR estimates suggested that the specification with unrestricted intercept and trend was the most appropriate. Moreover, given that the data used were quarterly, the variant with seasonal dummies was also preferred. There is variability in the suggested number of cointegrating relationships across the range of specifications used, and between the trace test and the maximal eigenvalue test used to ascertain this number. The surprise is that despite the results from the static cointegrating regressions and error correction models, which overwhelmingly point to no cointegration, all of the results in Table 5, except one, suggest at least one cointegrating vector. In the case of the preferred specification, the suggestion is of one cointegrating relationship, in contrast to the outcome produced by the Engle-Granger approach.

For the Finnish data, the summary results of the Johansen procedure on the full model are given in Table 6. There is similar variability in the number of cointegrating relationships suggested for the different specifications and tests to that noted for Denmark, though it is not quite as marked. The preferred specification is again that with unrestricted intercept, unrestricted trend and seasonal dummies, for which case the number of cointegrating relationships indicated is two, again in stark contrast to the earlier indications of no cointegration. As Johansen has pointed out, the interpretation of the findings for the Finnish data poses particular problems. Accordingly, two alternative reduced models for Finland were also investigated: one taking p_t to be $I(0)$ in the VAR analysis, and the other treating both p_t and i_t as $I(0)$. The summary results for these cases are given in Table 7 and Table 8, respectively. Table 7 contains consistent indications of a single cointegrating vector across all VAR specifications and tests, though once again this finding contradicts the indications from the AEG, CRDW and ECM tests. Slight variability in the results for different specifications and tests is seen in Table 8, but in this case no cointegration is suggested for the preferred specification. This finding conflicts with the corresponding AEG and CRDW results, which indicate a possibility of cointegration, but it is in agreement with the ECM test result.

The Johansen bias-correction factor was calculated only for the preferred VAR specification in the case of Denmark, and for the preferred specification of the full and the two reduced models in the case of Finland. Table 9 presents the Danish results. Although the correction factor relates only to the trace test, details of the maximum eigenvalue test are also given. The corresponding results for the full Finnish model and the two reduced versions are given in tables 10, 11 and 12, respectively. Interestingly, when the adjusted critical value is used for the trace test, the result for Denmark changes to one suggesting no cointegrating relationships, in accordance with the AEG, CRDW and ECM test findings. Thus there is conflict between the trace test and the

[4] The AIC and unadjusted likelihood-ratio test suggested a lag length of two. The choice of lag length one has the advantage of economizing on degrees of freedom.

maximum eigenvalue test in the case considered, the latter indicating one cointegrating relationship. The correction factors are close to unity for the Finland cases, probably due to the larger sample size. Even so, the outcome for the full Finnish model is similar to that for Denmark, the modified trace test indicating the reduced number of one cointegrating relationship, while the maximum eigenvalue test indicates two. However, the correction has no effect in the cases of the two reduced models. In particular, as the correction would increase the critical value of the trace statistic, and as the test statistic for the second reduced model already lies well below the uncorrected critical value, as can be seen from Table 12, the correction factor was not even computed for this final case. The conclusion suggested by the modified Johansen procedure remains that the number of cointegrating vectors is one and zero for the first and second reduced Finnish models, respectively.

It can be seen from these various results that the traditional analysis is somewhat confusing. Examination of the Danish data seems to suggest that all variables are $I(1)$ and, using the Engle-Granger two-step procedure, that cointegration does not hold and error correction models are not appropriate. Yet, using the original Johansen VAR approach, there are strong indications of cointegration, which are only challenged if a bias corrected trace test is undertaken. The Finnish data give rise to some similar findings, although in contrast to the Danish case, unit root tests suggest that some of the series are possibly not $I(1)$. When allowance is made for this possibility, the Engle-Granger approach marginally supports cointegration. However, when the Johansen technique is applied in this case, it gives contrary results, whether or not a modified trace test is used, indicating that there is no cointegration.

3.2.2 Fractional integration

Having raised concerns over the standard $I(1)/I(0)$ analysis, the next step is to consider the possibility of fractional integration. Table 13 gives the results of the fractional analysis for the Danish data. For each variable, a range of estimates of d is provided, as well as the results of the FDF and FADF tests. The corresponding results for the Finnish data are given in Table 14.

It can be seen from the results that there is little evidence in support of the Danish data being anything other than $I(1)$, which accords with the findings of the previous standard analysis. It is possible, if just the parametric estimators of d are considered, to argue that the Danish b_t variable is fractionally integrated, whereas for the Finnish data it would appear that three of the four variables are fractionally integrated, namely, m_t, p_t and i_t. It will be recalled that unit root tests decisively rejected the unit root null for the latter two variables. The results for Finland's y_t variable also give indications that it is fractionally integrated, but the FADF result in this case has the wrong logical sign. Overall, the investigation of fractional integration suggests that the Finnish data series are not generated by $I(1)$ processes but that the Danish data are.

3.2.3 The Hamilton approach

In light of the possibility that the emerging difficulties may be related to parameter instability, or some other type of nonlinearity, of what may be stationary data generating processes, simple recursive residual plots for the Danish and Finnish versions of model (19) were produced; these are depicted in figures 1 and 2, respectively, in Appendix A2. Guided by these graphs, simple Chow breakpoint tests for the Danish and Finnish models were implemented and the results of these are given in Table 15. Finally, Hamilton's random field approach was used to explore the likely form of the two models, and this leads to some interesting results. Clearly, the graphs and Chow tests provide strong initial evidence for structural instability in both models. Hamilton's LM test statistics for nonlinearity for the Danish and Finnish models were 15.338 and 123.810, respectively, which are significantly greater than the 5 per cent critical χ_1^2 figure of 3.84, again suggesting that the models should not be simple linear models. Detailed results from the Hamilton procedure are given in Table 16.

Given the earlier findings, the Hamilton results from the Danish data are rather disappointing, in so much as both σ and ζ estimates are not statistically significant on the basis of an asymptotic t-test. It could be argued, along the lines of Dahl and González-Rivera (2003), that this is due to nuisance parameter problems, given that under the null of linearity, the g_i parameters are unidentified. If the statistical insignificance of $\widetilde{\sigma}$ and $\widetilde{\zeta}$ is ignored, the significant coefficient of p_t in the linear and the nonlinear components of the Danish model strongly suggests that this inflation variable is the prime source of any parameter instability. This is in line with some of the results in Johansen and Juselius (1990).

In the case of Finland, the results in Table 16 are more satisfying. Both $\widetilde{\sigma}$ and $\widetilde{\zeta}$ are statistically significant, in agreement with the implied value of λ in the LM test, and suggesting that there is significant nonlinearity in the money demand relationship. In the Finnish case, it is the income variable, y_t, that proves significant in both the linear and nonlinear parts of the model and, therefore, that needs to be investigated further. The cross plots of m_t against p_t for Denmark, and of m_t against y_t for Finland, given in figures 3 and 4, respectively, in Appendix A2, hint at the possibility of a piecewise linear regression being an adequate model for the money demand relationships.

In the case of Denmark, such a model is

$$m_t = \alpha+\beta_1 y_t+\beta_2 p_t+\beta_3 (p_t - p_1) D_{1t}+\beta_4 (p_t - p_2) D_{2t}+\beta_5 i_t+\beta_6 b_t+\varepsilon_t, \quad (20)$$

$$\begin{aligned} p_1 = -0.44, \qquad & D_{1t} = 0, \quad p_t \leq p_1, \\ & D_{1t} = 1, \quad p_t > p_1, \\ p_2 = 0.26, \qquad & D_{2t} = 0, \quad p_t \leq p_2, \\ & D_{2t} = 1, \quad p_t > p_2. \end{aligned} \quad (21)$$

For Finland, an alternative model is

$$m_t = \alpha + \beta_1 i_t + \beta_2 p_t + \beta_3 y_t + \beta_4 (y_t - y_1) D_{1t} + \varepsilon_t, \quad (22)$$

$$y_1 = 4.3, \qquad \begin{array}{l} D_{1t} = 0, \quad y_t \leq y_1, \\ D_{1t} = 1, \quad y_t > y_1. \end{array} \tag{23}$$

In both equations (20) and (22), ε_t is a standard white noise error. The resulting OLS estimates, with standard errors given in parentheses, are

$$m_t = \underset{(0.67)}{6.66} + \underset{(0.11)}{0.93} y_t + \underset{(0.14)}{0.54} p_t + \underset{(0.17)}{1.25} (p_t - p_1) D_{1t} - \underset{(0.16)}{0.65} (p_t - p_2) D_{2t} + \underset{(0.58)}{0.61} i_t - \underset{(0.31)}{1.48} b_t, \tag{24}$$

and

$$m_t = \underset{(0.27)}{1.77} + \underset{(0.12)}{0.31} i_t - \underset{(0.46)}{0.32} p_t + \underset{(0.06)}{0.30} y_t + \underset{(0.09)}{0.88} (y_t - y_1) D_{1t} \tag{25}$$

for Denmark and Finland, respectively. In both cases the extra nonlinear terms are highly significant. Furthermore, the R^2 values are about 0.95 for both equations and the misspecification diagnostics for nonnormality, heteroscedasticity and functional form are also satisfactory. However, there are significant indications of first-order autocorrelation from the Durbin-Watson test, as well as fourth-order autocorrelation from the relevant Lagrange multiplier test.[5] Moreover, when the Hamilton test for nonlinearity is applied to these revised equations, the sample values of the LM statistics for the Danish and Finnish models are 42.987 and 18.354, respectively, which are still higher than the critical χ_1^2 value of 3.84. This finding contradicts the indications provided by the first test for nonlinearity (RESET), which suggests that a linear functional form is appropriate. Though the substantial fall in the value of the Hamilton test statistic for the Finnish data is encouraging, Hamilton's method suggests that both models are still not adequately specified. Clearly, were the application more than an illustrative case study, further work would be required to discover better nonlinear specifications for the money demand functions.

4 Summary and Conclusions

This paper has drawn attention to some of the pitfalls involved in using the conventional $I(1)/I(0)$ framework for economic and financial modelling of time-series data, an approach involving well-known unit root tests and the cointegration testing and modelling procedures of Engle and Granger (1987), and Johansen (1988), that has been applied widely during the last decade or so. The practical difficulties of untangling the issues of stationarity, fractional integration, nonlinearity, and parameter instability have been highlighted. In addition, the paper has discussed some of the recent research directed at resolving these problems and providing alternative, or at least complementary, approaches to modelling. Brief accounts have been given of the theory underlying fractional integration and long memory models, and of the estimation and testing methods in the random field regression approach proposed by Hamilton (2001). Some guidance has also been provided on the several methods of estimating and testing the order of fractional integration

[5] The detailed misspecification diagnostic results are available on request from the authors.

and the software necessary for the implementation of these and the Hamilton method.

A key element in the paper has been the presentation of a case study intended to illustrate the application of these newer techniques and contrast their findings with those of the standard cointegration modelling approach. The study used the data previously analyzed by Johansen and Juselius (1990) in connection with demand for money functions in Denmark and Finland. The results obtained from the various techniques exemplify the problems with the standard approach and the alternative conclusions that might be reached by using different techniques. The findings, using the standard approach, were as follows.

- Though ADF tests, implemented using the procedure of Dolado, et al. (1990), appear to suggest unit roots for most variables, they are sensitive to the specification of the test equation and the information criterion used to choose lag length in the case of some variables, especially for Finland.

- When the matter of unit roots was explored further, using the EPS, KPSS and NP tests, unit roots for the Danish variables tended to be confirmed but not for the Finnish variables.

- Proceeding on the assumption that all variables are $I(1)$, the Engle-Granger two-step procedure does not support cointegration in general, a result that is confirmed by ECM tests conducted in an error-correction framework for the money demand relationship for each country. However, the Engle-Granger approach does suggest cointegration for the version of the Finland model that treats two of the variables, p_t and i_t, as $I(0)$.

- Using the Johansen approach without its small sample bias-correction factor, there is considerably stronger evidence of cointegration in the case of Denmark, though the number of cointegrating vectors suggested varies, depending on the VAR specification chosen. For the preferred VAR specification, one cointegrating vector is suggested for Denmark. The picture that emerges for Finland is similar, although for the version of the model that treats the p_t and i_t variables as $I(0)$, the Johansen method suggests no cointegration, contradicting the finding of the Engle-Granger procedure in this case.

- The Johansen correction factor has a marked effect on the result in the case of the small sample of data for Denmark, the modified trace test agreeing with the conclusion from the Engle-Granger procedure that there is no cointegrating demand for money relationship. However, it was noted that the modified trace test provides a different signal from the maximum eigenvalue test, which indicates cointegration. As might be expected, the Johansen correction has no effect on the findings for Finland, which are based on a much larger sample.

These results are puzzling, not withstanding the relatively small size of the Danish sample used and the known low power of unit root tests. In particular, the contradictory results from the Engle-Granger and Johansen procedures concerning the existence of cointegrating relationships, in the case of both countries, is curious.

Checking for fractional integration by means of a range of estimators of the fractional integration parameter, as well as the new FDF and FADF tests of Dolado, et al. (2002), confirms the $I(1)$ nature of the Danish variables and the lack of a unit root for the variables in the case of Finland. It is difficult to say why the bias-corrected Johansen technique fails to find cointegration in the former case and yet suggests it in the latter.

Assuming that the Finnish data are not $I(1)$, and hence can not be simply cointegrated, what type of model is appropriate? The possibility of stationarity with regime shifts or some other kind of nonlinearity arises. This was explored, for both countries in fact, by means of recursive residual analysis and Chow tests, as well as by the Hamilton procedure, which is more appropriate for general, unknown forms of nonlinearity. These methods produce strong evidence of structural change/nonlinearity, if underlying stationarity is entertained. However, an attempt to re-specify the money demand equations as piecewise linear regressions, which was suggested by examination of the data, was not very successful. Clearly, further work would be necessary to find an adequate nonlinear functional form, were this alternative approach to be the preferred one.

In conclusion, the messages from this study appear to be that, first, standard $I(1)/I(0)$ modelling strategies for economic and financial time series are fraught with dangers. Secondly, complementary procedures designed to investigate the possibilities of fractional integration and nonlinearity are available and relatively easy to implement. Thirdly, fractional integration analysis may confirm the existence of unit roots, but may also suggest fractional integration of different degrees for different variables. This is a complicated situation that raises challenges for modelling. Fourthly, and recalling that unit root tests may often indicate that a unit root exists when a series is stationary but subject to level shifts, a general analysis of nonlinearity, such as that offered by the Hamilton procedure, may be an attractive option that can lead to acceptable alternative models. The moral would seem to be that reliance on any one approach may not be a sensible practice in applied work, and that practitioners would be well advised to consider using a range of alternative methods and selecting models according to the balance of the wider body of evidence produced.

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A Appendices

A.1 Tables

Table 1: Unit Root Tests, Denmark ($T = 55$)

Variable	Test Statistic probability†	Constant	Trend	Lag Length using SIC
m_t	1.123 (0.930)	Not Significant	Not Significant	2
y_t	0.776 (0.878)	Not Significant	Not Significant	0
p_t	0.437 (0.664)‡	Significant	Significant	0
i_t	−0.616 (0.446)	Not Significant	Not Significant	0
b_t	−0.982 (0.288)	Not Significant	Not Significant	1

†MacKinnon (1996) one-sided p-values.
‡normal probability.

Table 2: Unit Root Tests, Finland ($T = 106$)

Variable	Test Statistic probability†	Constant	Trend	Lag Length using SIC
m_t	1.720 (0.979)	Not Significant	Not Significant	4
y_t	−1.951 (0.054)‡	Significant	Not Significant	4
p_t	−4.200 (0.006)	n/a	n/a	1
i_t	−4.874 (0.001)	n/a	n/a	0

†MacKinnon (1996) one-sided p-values.
‡normal probability.
n/a denotes not applicable.

Table 3: Engle-Granger Levels Models

	Coefficients (*t*-statistics)			
Variable	Denmark ($T = 55$)	Finland ($T = 106$)		
	Full	Full	Reduced 1	Reduced 2
c	4.472 (5.178)	-0.841 (-6.552)	-0.784 (-6.104)	-0.766 (-6.448)
y_t	1.283 (9.135)	0.928 (31.988)	0.926 (31.300)	0.921 (35.021)
p_t	0.004 (0.122)	-0.366 (-0.552)	-0.254 (-0.377)	-
i_t	0.569 (0.705)	0.375 (2.285)	-	-
b_t	-2.601 (-7.370)	-	-	-
R^2	0.926	0.926	0.922	0.922
CRDW [critical values]	0.737 [1.19]	0.418 [0.58]	0.399 [0.48]	0.398 [0.38]
AEG test on residuals [critical value]	-3.301 [-4.694]	-3.541 [-4.204]	-3.473 [-3.824]	-3.461 [-3.395]

Note: 5 per cent AEG and CRDW critical values.

Table 4: Error Correction Models

	COEFFICIENTS (*t*-statistics)			
VARIABLE	DENMARK ($T = 55$)	FINLAND ($T = 106$)		
	Full	Full	Reduced 1	Reduced 2
c	0.008 (0.657)	0.004 (0.852)	0.004 (0.795)	0.004 (0.807)
Δm_{t-1}	−0.088 (−0.710)	−0.166 (−1.873)	−0.158 (−1.779)	−0.161 (−1.806)
Δm_{t-4}	0.387 (2.861)	0.093 (0.978)	0.099 (1.028)	0.098 (1.021)
Δy_t	0.497 (2.815)	0.502 (4.412)	0.502 (4.367)	0.501 (4.371)
Δp_t	−0.233 (−0.449)	−0.416 (−1.218)	−0.392 (−1.140)	−0.372 (−1.084)
Δi_t	−1.137 (−1.700)	0.247 (2.058)	0.222 (1.817)	0.221 (1.814)
Δb_t	−0.860 (−1.744)	-	-	-
ECM	−0.009 (−0.082)	0.175 (2.852)	0.154 (2.571)	0.157 (2.636)
R^2	0.498	0.532	0.525	0.526
DW	1.988	1.807	1.815	1.807
Serial correlation $\chi^2(4)$	5.119 [0.275]	6.460 [0.167]	5.187 [0.269]	5.553 [0.235]
Functional form $\chi^2(1)$	1.443 [0.230]	0.322 [0.570]	0.372 [0.542]	0.371 [0.542]
Normality $\chi^2(2)$	3.457 [0.178]	1.114 [0.573]	1.202 [0.548]	1.264 [0.532]
Heteroscedasticity $\chi^2(1)$	0.099 [0.753]	3.804 [0.051]	3.929 [0.047]	4.072 [0.044]

Note: For diagnostics, p-values in square brackets.

Table 5: Number of Cointegrating Relations by Model, Danish Data

Test Type	no inpts no trends		rest'd inpts no trends		unrest'd inpts no trends		unrest'd inpts rest'd trends		unrest'd inpts unrest'd trends	
	0.05 and 0.10 significance levels, excluding seasonal dummies									
Trace	3	4	3	3	3	4	3	3	1	2
Max-Eig.	1	2	3	3	3	4	3	3	0	1
	0.05 and 0.10 significance levels, including seasonal dummies									
Trace	4	4	3	3	3	3	2	3	1	2
Max-Eig.	1	2	2	3	2	3	2	3	1	1

Table 6: Number of Cointegrating Relations by Model, Finnish Data 1

Test Type	no inpts no trends		rest'd inpts no trends		unrest'd inpts no trends		unrest'd inpts rest'd trends		unrest'd inpts unrest'd trends	
	0.05 and 0.10 significance levels, excluding seasonal dummies									
Trace	3	4	2	4	2	2	2	2	2	2
Max-Eig.	2	4	2	2	2	2	2	2	2	2
	0.05 and 0.10 significance levels, including seasonal dummies									
Trace	3	4	3	4	2	2	2	2	2	2
Max-Eig.	3	4	2	4	2	2	2	2	2	2

Table 7: Number of Cointegrating Relations by Model, Finnish Data 2

Test Type	no inpts no trends		rest'd inpts no trends		unrest'd inpts no trends		unrest'd inpts rest'd trends		unrest'd inpts unrest'd trends	
	0.05 and 0.10 significance levels, excluding seasonal dummies									
Trace	1	1	1	1	1	1	1	1	1	1
Max-Eig.	1	1	1	1	1	1	1	1	1	1
	0.05 and 0.10 significance levels, including seasonal dummies									
Trace	1	1	1	1	1	1	1	1	1	1
Max-Eig.	1	1	1	1	1	1	1	1	1	1

Table 8: Number of Cointegrating Relations by Model, Finnish Data 3

Test Type	no inpts no trends		rest'd inpts no trends		unrest'd inpts no trends		unrest'd inpts rest'd trends		unrest'd inpts unrest'd trends	
	0.05 and 0.10 significance levels, excluding seasonal dummies									
Trace	0	1	0	1	0	0	0	0	0	0
Max-Eig.	0	1	0	0	0	0	0	0	0	0
	0.05 and 0.10 significance levels, including seasonal dummies									
Trace	1	1	1	2	0	0	0	0	0	0
Max-Eig.	1	1	1	1	0	0	0	0	0	0

Table 9: Johansen Results for Danish Data plus Modified Critical Values

Unrestricted Cointegration Rank Test (Trace)					
Hypotheses		Trace Statistic	0.05 Critical Value	0.10 Critical Value	Modified 0.05 Critical Value
$r = 0$	$r \geq 1$	111.641	82.230	77.550	112.803
$r \leq 1$	$r \geq 2$	55.809	58.930	55.010	-
$r \leq 2$	$r \geq 3$	29.843	39.330	36.280	-
$r \leq 3$	$r \geq 4$	7.640	23.830	21.230	-
$r \leq 4$	$r = 5$	0.094	11.540	9.750	-

Unrestricted Cointegration Rank Test (Maximum Eigenvalue)				
Hypotheses		Trace Statistic	0.05 Critical Value	0.10 Critical Value
$r = 0$	$r = 1$	55.831	37.070	34.160
$r \leq 1$	$r = 2$	25.966	31.000	28.320
$r \leq 2$	$r = 3$	22.203	24.350	22.260
$r \leq 3$	$r = 4$	7.546	18.330	16.280
$r \leq 4$	$r = 5$	0.094	11.540	9.750

Note: The correction factor is 1.372.

Table 10: Johansen Results for Finnish Data 1 plus Modified Critical Values

Unrestricted Cointegration Rank Test (Trace)

Hypotheses		Trace Statistic	0.05 Critical Value	0.10 Critical Value	Modified 0.05 Critical Value
$r = 0$	$r \geq 1$	85.122	58.930	55.010	63.998
$r \leq 1$	$r \geq 2$	42.621	39.330	36.280	42.712
$r \leq 2$	$r \geq 3$	12.207	23.830	21.230	-
$r \leq 3$	$r \geq 4$	2.247	11.540	9.750	-

Unrestricted Cointegration Rank Test (Maximum Eigenvalue)

Hypotheses		Trace Statistic	0.05 Critical Value	0.10 Critical Value
$r = 0$	$r = 1$	42.501	31.000	28.320
$r \leq 1$	$r = 2$	30.414	24.350	22.260
$r \leq 2$	$r = 3$	9.961	18.330	16.280
$r \leq 3$	$r = 4$	2.247	11.540	9.750

Note: The correction factor is 1.086.

Table 11: Johansen Results for Finnish Data 2 plus Modified Critical Values

Unrestricted Cointegration Rank Test (Trace)					
Hypotheses		Trace Statistic	0.05 Critical Value	0.10 Critical Value	Modified 0.05 Critical Value
$r = 0$	$r \geq 1$	43.798	39.330	36.280	40.313
$r \leq 1$	$r \geq 2$	10.795	23.830	21.230	-
$r \leq 2$	$r \geq 3$	2.052	11.540	9.750	-

Unrestricted Cointegration Rank Test (Maximum Eigenvalue)				
Hypotheses		Trace Statistic	0.05 Critical Value	0.10 Critical Value
$r = 0$	$r = 1$	33.004	24.350	22.260
$r \leq 1$	$r = 2$	8.742	18.330	16.280
$r \leq 2$	$r = 3$	2.052	11.540	9.750

Note: The correction factor is 1.025.

Table 12: Johansen Results for Finnish Data 3 plus Modified Critical Values

Unrestricted Cointegration Rank Test (Trace)				
Hypotheses		Trace Statistic	0.05 Critical Value	0.10 Critical Value
$r=0$	$r\geq 1$	8.827	23.830	21.230
$r\leq 1$	$r\geq 2$	0.300	11.540	9.750

Unrestricted Cointegration Rank Test (Maximum Eigenvalue)				
Hypotheses		Trace Statistic	0.05 Critical Value	0.10 Critical Value
$r=0$	$r=1$	8.527	18.330	16.280
$r\leq 1$	$r=2$	0.300	11.540	9.750

Table 13: Fractional Integration Analysis, Danish Data

	MLE	NLS	GPH	GSP	FDF†	FADF†
m_t	1.159 (0.123)	1.176 (0.133)	1.168 (0.171)	0.993 (0.096)	-0.850	-1.490
y_t	0.360 (0.281)	0.577 (0.276)	1.23 (0.171)	1.08 (0.096)	-0.076	2.267
p_t	0.741 (0.301)	0.674 (0.231)	0.994 (0.171)	0.870 (0.096)	0.077	-1.569
i_t	0.574 (0.275)	0.521 (0.260)	1.171 (0.171)	1.084 (0.962)	-0.223	-1.125
b_t	0.738 (0.278)	0.727 (0.218)	1.377 (0.171)	1.275 (0.096)	2.339	-2.191

†Based on the MLE estimator of d.

Table 14: Fractional Integration Analysis, Finish Data

	MLE	NLS	GPH	GSP	FDF†	FADF†
m_t	0.778 (0.090)	0.762 (0.090)	0.830 (0.112)	0.590 (0.069)	-2.94	-2.046
y_t	0.559 (0.084)	0.570 (0.086)	0.745 (0.113)	0.523 (0.693)	1.250	8.502
p_t	0.236 (0.099)	0.210 (0.096)	0.410 (0.114)	0.394 (0.113)	-6.45	-2.63
i_t	0.621 (0.108)	0.622 (0.103)	0.759 (0.112)	0.796 (0.069)	-2.73	-3.54

†Based on the MLE estimator of d.

Table 15: Simple Chow Breakpoint Tests

	Denmark	Finland
F-statistic	10.127 (0.000)	15.205 (0.000)
Log-likelihood ratio	69.400 (0.000)	70.273 (0.000)

Note: Two breakpoints each model.

Table 16: Hamilton Analysis

	Denmark Estimates (standard error)	Finland Estimates (standard error)
Linear		
c	$-$ 7.338 (1.142)	-0.554 (0.348)
y_t	0.781 (0.190)	0.877 (0.079)
p_t	0.129 (0.061)	-0.826 (0.456)
i_t	-0.066 (0.063)	0.133 (0.171)
b_t	-0.111 (0.039)	
Non-linear		
σ	0.009 (0.006)	0.050 (0.005)
ζ	5.376 (4.014)	1.289 (0.311)
y_t	3.412 (2.335)	4.791 (0.748)
p_t	6.490 (1.394)	0.009 (0.360)
i_t	-0.00002 (0.510)	2.238 (2.167)
b_t	0.000003 (0.569)	

A.2 Figures

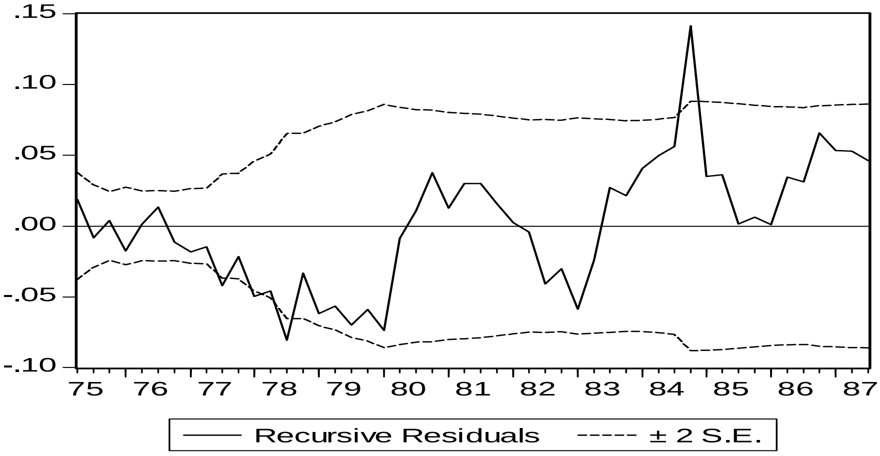

Figure 1: Recursive residuals plot, Denmark

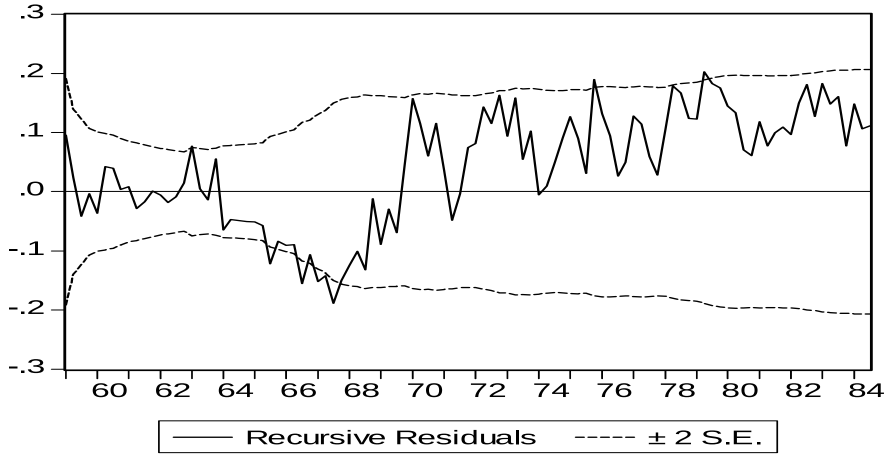

Figure 2: Recursive residuals plot, Finland

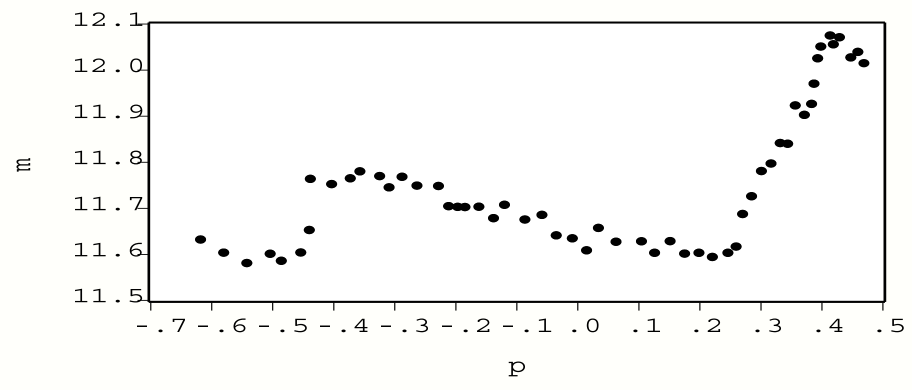

Figure 3: m_t against p_t, Denmark

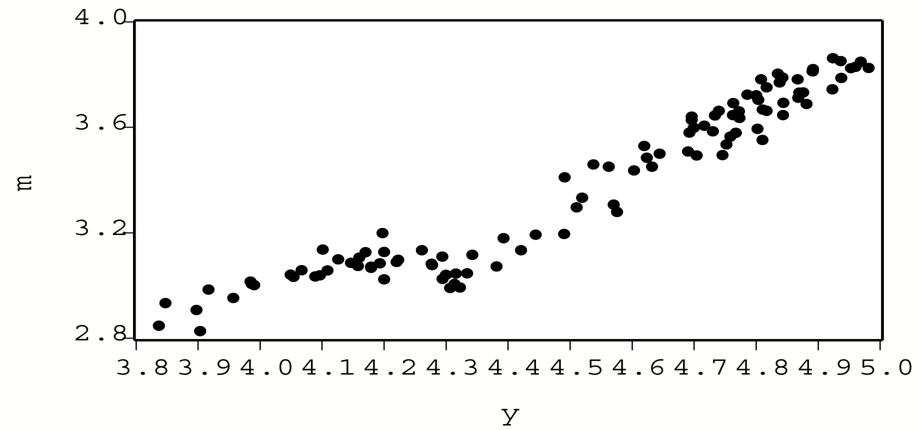

Figure 4: m_t against y_t, Finland