Constrained Optimization

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- **Reading**

1. Chapters 12.1-12.3 and 12.5, 13.5, of CW
2. Chapter 15, of PR

- **Plan**

1. Unconstrained versus constrained optimization problems
2. Lagrangian formulation, second-order conditions, bordered Hessian matrix
3. Envelope theorem

Constrained Optimization: Examples

- Until now, we have consider unconstrained problems. Usually, economic agents face natural constraints.
- *Consumer's problem*: Suppose that a consumer has a utility function $U(x, y) = x^{0.5}y^{0.5}$, the price of x is \$2, the price of y is \$3 and the consumer has \$100 in income. How much of the two goods should the consumer purchase to maximize her utility?
- *Firm's problem*: Suppose that a firm's production function is given by $q = K^{0.5}L^{0.5}$, the price of capital is \$2 and the price of labour is \$5. What is the least cost way for the firm to produce 100 units of output?

Relevant Mathematics

- Both of the above problems have a common mathematical structure:

$$\max_{x_1,\ldots,x_n} f(x_1, x_2, \ldots, x_n) \text{ subject to } g(x_1, x_2, \ldots, x_n) = 0$$

- We say that $f(x_1, x_2, \ldots, x_n)$ is the objective function, $g(x_1, x_2, \ldots, x_n)$ is the constraint, and $x_1, \ldots, x_n$ are the choice variables.
- It is also possible that instead of maximizing $f(x_1, \ldots, x_n)$ we could be minimizing $f(x_1, \ldots, x_n)$.
- $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_n)$ are just regular functions.

More Specific Examples

- The above problems can be translated into the above mathematical framework as
- Utility maximization:

$$\max_{(x,y)} x^{0.5}y^{0.5} \text{ subject to } 2x + 3y - 100 = 0$$

- Cost minimization:

$$\min_{(K,L)} K^{0.5}L^{0.5} \text{ subject to } 2K + 5L - 100 = 0$$

- That is, in case one, $U(x, y) = x^{0.5}y^{0.5}$, and in case two, $F\left(K, L\right) = K^{0.5}L^{0.5}$.

One Approach (Direct Substitution)

- When the constraint(s) are **equalities**, we can convert the problem from a constrained optimization to an unconstrained optimization problem by substituting for some of the variables.
- In the consumer's problem, we have $2x + 3y = 100$, so $x = 50 - (3/2)y$. We can use this relation to substitute for x in the utility function which gives

$$\max_{(y)} U(x, y) = x^{0.5}y^{0.5} = (50 - (3/2)y)^{0.5}\, y^{0.5}$$

- This is now a function of just y and we can now maximize this function with respect to y. It is important to observe that this is an unconstrained optimization problem since we have incorporated the constraint by substituting for x.

Utility Maximization with Direct Substitution

- The first order conditions for the maximization of $(50-(3/2)y)^{0.5}\,y^{0.5}$ gives us the following.

$$\frac{1}{2}\left[-\frac{3}{2}\right]\left[50-\frac{3}{2}y\right]^{-0.5}y^{0.5}+\frac{1}{2}\left[50-\frac{3}{2}y\right]^{0.5}y^{-0.5}=0$$

- Solving this gives

$$-\frac{3}{2}\left[50-\frac{3}{2}y\right]^{-0.5}y^{0.5}=\left[50-\frac{3}{2}y\right]^{0.5}y^{-0.5}$$
$$y=50/3$$

- Since $x=50-(3/2)y$, we also have,

$$\begin{aligned} x &= 50-(3/2)50/3 \\ &= 25 \end{aligned}$$

Cost Minimization with Direct Substitution

- For the firm's minimization problem, we can proceed similarly:
- The constraint gives us $K^{1/2}L^{1/2} = 100$ or $K = 10000/L$. Therefore, the problem is the following.

$$\min_{(L)} 20000/L + 5L$$

- This can be minimized easily with respect to L, and then the corresponding K found easily.
- That is, $-20000L^{-2} + 5 = 0 \Rightarrow L = (5/20000)^{-1/2}$ and $K = \sqrt[2]{4000} \approx 63.2$.

The Lagrangian Approach

- Two reasons for an alternative approach:

1. In some cases, we cannot use substitution easily: for instance, suppose the constraint is $x^4 + 5x^3y + y^2x + x^6 + 5 = 0$. Here, it is not possible to solve this equation to get x as a function of y or vice versa.
2. In many cases, the economic constraints are written in the form $g(x_1, \ldots, x_n) \leq 0$ or $g(x_1, \ldots, x_n) \geq 0$. While the Lagrangian technique can be modified to take care of such cases, the substitution technique cannot be modified, or can be modified only with some difficulty.

How does the Lagrangean technique work?

- Given a problem

$$\max f(x_1, \ldots, x_n) \text{ subject to } g(x_1, \ldots, x_n) = 0$$

- Write down the Lagrangian function

$$\mathcal{L}(x_1, \ldots, x_n, \lambda) = f(x_1, \ldots, x_n) + \lambda g(x_1, \ldots, x_n)$$

- Note that the Lagrangian is a function of $n+1$ variables: $(x_1, \ldots, x_n, \lambda)$. We then look for the *critical points* of the Lagrangian, that is, points where all the partial derivatives of the Lagrangian are zero.
- *Note that we are not trying to maximize or minimize the Lagrangian function.*

First Order conditions with Lagrangean technique

- Using a Lagrangian, we get $n + 1$ first order conditions:

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0, (i = 1, \ldots, n)$$
$$\text{and } \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

- Solving these equations will give us candidate solutions for the constrained optimization problem.
- Candidate solutions have the same status as the unconstrained case. That is, they need to be checked using the second-order conditions.

Utility Maximization Revisited

- In the consumer's problem, the Lagrangian function is

$$\mathcal{L}(x, y, \lambda) = x^{0.5}y^{0.5} + \lambda(100 - 2x - 3y)$$

- We get the three first order conditions and use these to solve for the equilibrium values of x and y.
- The first order conditions for optimization gives us:

$$\frac{\partial \mathcal{L}}{\partial x} = 0.5x^{-0.5}y^{0.5} - 2\lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = 0.5x^{0.5}y^{-0.5} - 3\lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = 100 - 2x - 3y = 0$$

Utility Maximization Revisited

- From the first two equations, we get

$$\frac{\partial \mathcal{L}}{\partial x} = 0.5x^{-0.5}y^{0.5} - 2\lambda = \frac{\partial \mathcal{L}}{\partial y} = 0.5x^{0.5}y^{-0.5} - 3\lambda$$

$$\Rightarrow \frac{x^{-0.5}y^{0.5}}{x^{0.5}y^{-0.5}} = \frac{2}{3} \text{ or } y = (2/3)x$$

- Substituting this into the third gives us,

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 100 - 2x - 3y100 - 2x - 3(2/3)x = 0$$

- So, $x = 25$ and therefore, $y = 50/3$. That is exactly what we got before.

Cost Minimization Revisited

- In the second case, we have the following Lagrangian:

$$\mathcal{L}(K, L, \lambda) = 2K + 5L + \lambda(100 - K^{0.5}L^{0.5})$$

- The first order conditions for optimization gives us:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial K} &= 2 - 0.5\lambda K^{-0.5}L^{0.5} = 0 \\
\frac{\partial \mathcal{L}}{\partial L} &= 5 - 0.5\lambda K^{0.5}L^{-0.5} = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= 100 - K^{0.5}L^{0.5} = 0
\end{aligned}$$

Cost Minimization Revisited

- Again, the first two conditions give us

$$\frac{2}{5} = \frac{K^{-0.5}L^{0.5}}{K^{0.5}L^{-0.5}}$$

- Or $L = (2/5)K$.
- Substituting into the third gives us,

$$K = \frac{100\sqrt{5}}{\sqrt{2}}, L = \frac{100\sqrt{2}}{\sqrt{5}}$$

- Earlier, we got $K = \sqrt[2]{4000} \approx 63.2$. Now we have, $\frac{100\sqrt{5}}{\sqrt{2}} \approx 63.2$.
- **Note:** we will check the second order conditions later on.

Observations

1. Note that the technique has been identical for both maximization and minimization problems. This means that the first order conditions identified so far are only necessary conditions and not sufficient conditions. We shall look at sufficient, or second order conditions later.
2. Note also that we did not compute λ. This is because our interest is in the values of x and y (or K and L). However, in some instances, it is useful to compute λ: this has an economic interpretation in terms of the *shadow price of the constraint*.

What do the first order conditions from the Lagrangean method say?

- Take the consumer's problem. If we divide the first two conditions, we get that
$$\frac{U_x}{U_y} = \frac{p_x}{p_y}$$
- This says that at the optimum point, the slope of the indifference curve must be equal to the slope of the budget line.
- Similarly, the first order conditions for the firm's cost minimization problem says that
$$\frac{F_K}{F_L} = \frac{r}{w}$$
- This has a similar interpretation.

More on the Lagrangian Approach

- Basically, the Lagrangian approach amounts to searching for points where:

1. The constraint is satisfied.
2. The constraint and the level curve of the objective function are tangent to one another.

- If we have more than two variables, then the same intuition can be extended. For instance, with three variables, the Lagrangian conditions will say:

1. The rate of substitution between any two variables along the objective function must equal the rate of substitution along the constraint.
2. The optimum point must be on the constraint.

Slightly More Complicated Example (I)

- Consider the problem

$$\max f(x, y) = x^2 + y^2 \text{ subject to } x^2 + xy + y^2 = 3$$

- We write the Lagrangian

$$\mathcal{L}(x, y, \lambda) = x^2 + y^2 + \lambda(3 - x^2 - xy - y^2)$$

- The first-order conditions are

$$\begin{aligned} \mathcal{L}_x &= 2x - \lambda(2x + y) = 0 \\ \mathcal{L}_y &= 2y - \lambda(x + 2y) = 0 \\ \mathcal{L}_\lambda &= 3 - x^2 - xy - y^2 = 0 \end{aligned}$$

Slightly More Complicated Example (II)

- Finding the solution to this set of equations is not too difficult, but one needs to be careful.
- From the first equation, *provided* $y \neq -2x$, we find,

$$\lambda = \frac{2x}{2x + y}$$

- Substituting this into the second equation gives

$$\begin{aligned} 2y &= \frac{2x}{2x+y}(x+2y) \\ \text{or } 4xy + 2y^2 &= 2x^2 + 4xy \\ \text{or } y^2 &= x^2 \end{aligned}$$

- The solution to $y^2 = x^2$ can involve two possibilities: (i) $y = x$ and (ii) $y = -x$.

Slightly More Complicated Example (III)

- Suppose $y = x$.
- Substituting for y in the third equation gives $3 - x^2 - x^2 - x^2 = 0$ or $3 - 3x^2 = 0$ or $3(1-x)(1+x) = 0$. Thus, we have that either $x = 1$ or $x = -1$. Since $y = x$, it follows that the only combinations of (x, y) which satisfy the first-order conditions and for which $y = x$ are $(1, 1)$ and $(-1, -1)$.
- Suppose $y = -x$.
- Again, we substitute for y in the third equation to get $3 - x^2 - x(-x) - (-x)^2 = 0$ or $3 - x^2 = 0$ or $(\sqrt{3} - x)(\sqrt{3} + x) = 0$ which gives $x = \sqrt{3}$ or $x = -\sqrt{3}$. Since $y = -x$, it follows that the only solutions to the first order conditions for which $y = -x$ are $(\sqrt{3}, -\sqrt{3})$ and $(-\sqrt{3}, \sqrt{3})$.

Slightly More Complicated Example (IV)

- Finally, we have to consider the possibility that $y = -2x$.
- In this case, the first equation implies $x = 0$ and hence, $y = 0$.
- However, then the third equation cannot be satisfied. Hence, it cannot be the case that $y = -2x$.
- We thus have four solutions to the first order conditions:

$$\begin{aligned} & (x, y) = (1, 1) \\ \text{or } & (x, y) = (-1, -1) \\ \text{or } & (x, y) = (\sqrt{3}, -\sqrt{3}) \\ \text{or } & (x, y) = (-\sqrt{3}, \sqrt{3}) \end{aligned}$$

Multiple Constraints

- If we have more than one constraint, then the same technique can be extended. For instance, if we have a consumer's problem

$$\max x^{0.5}y^{0.2}z^{0.2}$$

subject to,

$$100 - 2x - 3y = 0$$
$$20 - y - 4z = 0$$

- We proceed by creating the Lagrangian.

$$\mathcal{L}(x, y, z, \lambda_1, \lambda_2) = x^{0.5}y^{0.2}z^{0.2} + \lambda_1(100 - 2x - 3y) + \lambda_2(20 - y - 4z)$$

- We then look for points $(x, y, z, \lambda_1, \lambda_2)$ where the partial derivatives of the Lagrangian function are all zero. In this case, note that solving the first order conditions is not easy (so we won't look into this too much).

Economic Interpretation of the Lagrangian multiplier

- Suppose we have the problem,

$$\max f(x, y) \text{ subject to } g(x, y) = 0$$

- Suppose we now *relax* this constraint: instead of requiring $g(x, y) = 0$ we require $g(x, y) = \delta$ where δ is a small positive number.
- Clearly, since the constraint has been changed, the value of the objective function must change. The question is: by how much?
- The answer to this question is given by λ. For this reason, λ is referred to as the *shadow price of the constraint*. It tells us the rate at which the objective function increases if the constraint is changed by a small amount.

Economic Interpretation of the Lagrangian multiplier - Utility Maximization Case

- Consider the consumer's problem discussed earlier.
- We can compute $\lambda = \dfrac{1}{2\sqrt{6}}$. What does this mean? We had written the constraint as $2x + 3y - 100 = 0$.
- Change the constraint to $2x + 3y - 100 = \delta$ or $2x + 3y = 100 + \delta$, which means that we are giving the consumer an additional income of $\delta > 0$.
- The shadow price of the constraint $\lambda = \dfrac{1}{2\sqrt{6}}$ tells us that if we give a small amount of additional income to the consumer then his utility will go up by a factor of $\frac{1}{2\sqrt{6}}$.

Second Order Conditions

- As with the unconstrained case, we need to check the second order conditions to ensure we have a global optimum.
- The second order conditions for constrained optimization differ slightly from the usual conditions.
- Suppose we have the problem

$$\max f(x, y) \text{ or } \min f(x, y) \text{ subject to } g(x, y) = 0$$

- The Lagrangian is the following.

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

- We compute the *bordered Hessian matrix* at the critical point that we want to check.

Bordered Hessian

- The bordered Hessian is the 'usual Hessian', bordered by the derivatives of the constraint with respect to the endogenous variables, here x and y. That is,

$$H^B = \begin{bmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ g_y & \mathcal{L}_{xy} & \mathcal{L}_{yy} \end{bmatrix}$$

- The second order conditions state

1. If (x^*, y^*, λ^*) corresponds to a constrained maximum, then $|H^B|$ evaluated at (x^*, y^*, λ^*) must be positive.
2. If (x^*, y^*, λ^*) corresponds to a constrained minimum, then $|H^B|$ evaluated at (x^*, y^*, λ^*) must be negative.

Utility Maximization and the Bordered Hessian

- In the consumer's problem, we have,

$$\max_{(x,y)} x^{0.5}y^{0.5} \text{ subject to } 2x + 3y - 100 = 0$$

- The corresponding bordered Hessian matrix is,

$$H^B = \begin{bmatrix} 0 & -2 & -3 \\ -2 & -0.25x^{-1.5}y^{0.5} & 0.25x^{-0.5}y^{-0.5} \\ -3 & 0.25x^{-0.5}y^{-0.5} & 0.25x^{0.5}y^{-1.5} \end{bmatrix}$$

- Previously, our candidate solutions were $x = 25 > 0$ and $y = 50/3 > 0$.
- We need to check $|H^B|$ is positive for this solution. We can verify this without doing any computation.

Utility Maximization and the Bordered Hessian

- We need the determinant of the bordered Hessian.
- Expanding for the determinant about the first row, we have,

$$\begin{aligned} |H^B| &= (-1)^3 \times (-2) \times \begin{vmatrix} -2 & 0.25x^{-0.5}y^{0.5} \\ -3 & -0.25x^{0.5}y^{-1.5} \end{vmatrix} \\ &+ (-1)^4 \times (-3) \times \begin{vmatrix} -2 & -0.25x^{-1.5}y^{0.5} \\ -3 & 0.25x^{-0.5}y^{-0.5} \end{vmatrix} \end{aligned}$$

- Note that the first determinant is positive and the second negative since $x > 0, y > 0$ and hence, $|H^B|$ is positive.
- This shows that $(x, y) = (25, 50/3)$ is a local maximum.

Cost Minimization and the Bordered Hessian

- Recall the firms minimization problem. Using a Lagrangian,

$$\mathcal{L}(K, L, \lambda) = 2K + 5L + \lambda(100 - K^{0.5}L^{0.5})$$

- In the firm's minimization problem, the bordered Hessian matrix is,

$$H^B = \begin{bmatrix} 0 & -0.5K^{-0.5}L^{0.5} & -0.5K^{0.5}L^{-0.5} \\ -0.5K^{-0.5}L^{0.5} & 0.25\lambda K^{-1.5}L^{0.5} & -0.25\lambda K^{-0.5}y^{-0.5} \\ -0.5K^{0.5}L^{-0.5} & -0.25\lambda K^{-0.5}L^{-0.5} & 0.25\lambda K^{0.5}L^{-1.5} \end{bmatrix}$$

- Note that H^B contains λ and so we need to find the value of λ. From the first first-order condition

$$\lambda = \frac{2}{0.5K^{-0.5}L^{0.5}} = \frac{4\sqrt{5}}{\sqrt{2}}$$

Cost Minimization and the Bordered Hessian

- The painful way of proceeding is to substitute,

$$K = \frac{100\sqrt{5}}{\sqrt{2}}, L = \frac{100\sqrt{2}}{\sqrt{5}}, \lambda = \frac{4\sqrt{5}}{\sqrt{2}}$$

  into H^B

- Then we have to verify that its determinant is negative; again, there is a less painful way. We can expand the first row of the bordered Hessian.

Cost Minimization and the Bordered Hessian

- Expanding for the determinant about the first row, we have,

$$|H^B| = 0.5K^{-0.5}L^{0.5} \begin{vmatrix} -0.5K^{-0.5}L^{0.5} & -0.25\lambda K^{-0.5}L^{-0.5} \\ -0.5K^{0.5}L^{-0.5} & 0.25\lambda K^{0.5}L^{-1.5} \end{vmatrix}$$
$$- 0.5K^{0.5}L^{-0.5} \begin{vmatrix} -0.5K^{-0.5}L^{0.5} & 0.25\lambda K^{-1.5}L^{0.5} \\ -0.5K^{0.5}L^{-0.5} & -0.25\lambda K^{-0.5}L^{-0.5} \end{vmatrix}$$

- Using the fact that $\lambda > 0$, we can conclude that the first determinant is negative while the second is positive.
- Hence, $|H^B|$ is negative which shows that the solution to the first order conditions constitute a local minima.

The Envelope Theorem

- Suppose we have the unconstrained optimization problem $\max f(x, y, \alpha)$ where α is some exogenous variable.
- Suppose that $(x^*(\alpha), y^*(\alpha))$ solves this optimization problem. (Note that the solution will depend upon α.)
- The *Value function* for this problem is derived by substituting $(x^*(\alpha), y^*(\alpha))$ into the objective function:

$$V(\alpha) = f(x^*(\alpha), y^*(\alpha), \alpha)$$

- Notice that the value function is a function of the parameter α. Notice also that the value function depends on α in two different ways.

1. Direct dependence.
2. Indirect dependence through $x^*(\alpha)$ and $y^*(\alpha)$.

Example: Calculating the Value Function

- Consider the unconstrained problem,

$$\max_{(x_1,x_2)} 4x_1 + \alpha x_2 - x_1^2 - x_2^2 + x_1 x_2$$

- The first order conditions are the following,

$$4 - 2x_1 + x_2 = 0$$

$$\alpha - 2x_2 + x_1 = 0$$

- Solving this gives,

$$x_1^* = \frac{8+\alpha}{3},\ x_2^* = \frac{2\alpha+4}{3}$$

- You should check the second order conditions hold so that this candidate solution is a global maximum.
- What we now do is substitute these conditions back into the objective function.

Example: Calculating the Value Function

- The Value function is,

$$
\begin{aligned}
V(\alpha) &= 4x_1^* + \alpha x_2^* - (x_1^*)^2 - (x_2^*)^2 + x_1^* x_2^* \\
&= 4\frac{8+\alpha}{3} + \alpha\frac{2\alpha+4}{3} - \left[\frac{8+\alpha}{3}\right]^2 \\
&\quad - \left[\frac{2\alpha+4}{3}\right]^2 + \frac{(8+\alpha)(2\alpha+4)}{9}
\end{aligned}
$$

- We are interested in knowing how the value function changes when α changes.
- Recall that α is some exogenous variable that affects our decisions.

Value Function

- Since we are interested in knowing how the value function changes when α changes, consider the most general case.

$$V(\alpha) = f(x^*(\alpha), y^*(\alpha), \alpha)$$

- When we differentiate this, we get,

$$\frac{dV(\alpha)}{d\alpha} = \frac{\partial f}{\partial x}\frac{\partial x^*(\alpha)}{\partial \alpha} + \frac{\partial f}{\partial y}\frac{\partial y^*(\alpha)}{\partial \alpha} + \frac{\partial f}{\partial \alpha}$$

- **Note:** the partial derivatives of f are evaluated at the solution $(x^*(\alpha), y^*(\alpha))$.

Envelope Theorem

- Now note that at the optimum (assuming we have an interior solution), it must be the case that $\partial f / \partial x = 0$ and $\partial f / \partial y = 0$.
- Hence, the first two terms drop out and we have

$$\frac{dV(\alpha)}{d\alpha} = \frac{\partial f}{\partial \alpha}$$

- The partial derivative is evaluated at the point $(x^*(\alpha), y^*(\alpha))$.
- This result which is called the **Envelope Theorem** says in words: "The total derivative of the value function with respect to the parameter α is the same as the partial derivative of the objective function evaluated at the optimal point."

Envelope Theorem Example

- To see the Envelope Theorem in operation, consider the derivative of the value function for the maximization problem considered earlier.
- Since $f(x, y, \alpha) = 4x_1 + \alpha x_2 - x_1^2 - x_2^2 + x_1 x_2$. Hence $\partial f / \partial \alpha = x_2$.
- By the Envelope theorem, the derivative of the value function is the same as the partial derivative of f evaluated at the optimal point $(x^*(\alpha), y^*(\alpha))$. Therefore, constants.
- Hence,

$$\frac{dV(\alpha)}{d\alpha} = x_2^* = \frac{2\alpha + 4}{3}$$

- Now consider the constrained case. We can basically do the same as before.
- Consider the problem,

$$\max f(x, y, \alpha) \text{ subject to } g(x, y, \alpha) = 0$$

- The Lagrangian for this problem is,

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda g(x, y, \alpha)$$

- Suppose that $(x^*(\alpha), y^*(\alpha), \lambda^*(\alpha))$ solves the constrained optimization problem.
- The Value function for this problem is defined as,

$$V(\alpha) = f(x^*(\alpha), y^*(\alpha), \alpha)$$

The Envelope Theorem: Constrained optimization

- Let us write the Value function as,

$$V(\alpha) = f(x^*(\alpha), y^*(\alpha), \alpha) + \lambda^*(\alpha) g(x^*(\alpha), y^*(\alpha), \alpha)$$

- Differentiating with respect to α gives,

$$\begin{aligned}\frac{dV(\alpha)}{d\alpha} = &\frac{\partial f}{\partial x}\frac{dx^*}{d\alpha} + \frac{\partial f}{\partial y}\frac{dy^*}{d\alpha} + \frac{\partial f}{\partial \alpha} \\ &+ g(x^*(\alpha), y^*(\alpha), \alpha)\frac{d\lambda^*}{d\alpha} + \lambda^*(\alpha)\left[\frac{\partial g}{\partial x}\frac{dx^*}{d\alpha} + \frac{\partial g}{\partial y}\frac{dy^*}{d\alpha} + \frac{\partial g}{\partial \alpha}\right]\end{aligned}$$

The Envelope Theorem: Constrained optimization

- This can be written as,

$$\frac{dV(\alpha)}{d\alpha} = \left[\frac{\partial f}{\partial x} + \lambda^* \frac{\partial g}{\partial x}\right] \frac{dx^*}{d\alpha} + \left[\frac{\partial f}{\partial y} + \lambda^* \frac{\partial g}{\partial y}\right] \frac{dy^*}{d\alpha}$$
$$+ g(x^*(\alpha), y^*(\alpha), \alpha) \frac{d\lambda^*}{d\alpha} + \frac{\partial f}{\partial \alpha} + \lambda^* \frac{\partial g}{\partial \alpha}$$

- Note that the first two terms on the right hand side drop out because (x^*, y^*, λ^*) must satisfy the necessary conditions for constrained optimization.
- The third term drops out because $g(x^*(\alpha), y^*(\alpha), \alpha) = 0$.

The Envelope Theorem: Constrained optimization

- We are left with the following:

$$\frac{dV(\alpha)}{d\alpha} = \frac{\partial f}{\partial \alpha} + \lambda^* \frac{\partial g}{\partial \alpha} = \frac{\partial \mathcal{L}}{\partial \alpha}(x^*, y^*, \lambda^*)$$

- In words, the derivative of the value function with respect to the parameter α is the partial derivative of the Lagrangian function with respect to α evaluated at (x^*, y^*, λ^*).
- We can apply this reasoning to the consumer and firm problems. This makes it a very powerful tool.

Consumer Optimization, Again

- Consider a more general version of the consumer's problem:

$$\max U(x, y) = x^{0.5}y^{0.5} \text{ subject to } M - px - qy = 0$$

- $M > 0$ is the consumer's income, $p > 0$ is the price of good x and $q > 0$ the price of good y.
- By forming the Lagrangian, one can compute (and verify using the second-order conditions) that the optimal solution is,

$$x^* = \frac{M}{2p}, y^* = \frac{M}{2q}, \lambda^* = \frac{1}{2\sqrt{pq}}$$

Value Function for the General Consumer Problem

- The value function in this case is,

$$V(M, p, q) = (x^*)^{0.5} (y^*)^{0.5} = \left(\frac{M}{2p}\right)^{0.5} \left(\frac{M}{2q}\right)^{0.5} = \frac{M}{2\sqrt{pq}}$$

- Note that the value function in this case depends on three parameters: (M, p, q). However, we can still apply the Envelope theorem.
- If, for instance, we want to know how the Value function changes when M changes, we simply treat p and q as constants.

Value Function for the General Consumer Problem

- We can now verify the validity of the Envelope theorem.
- The Lagrangian for this problem is

$$\mathcal{L} = (x)^{0.5}(y)^{0.5} + \lambda[M - px - qy]$$

- By the Envelope Theorem, we have

$$\frac{\partial V}{\partial M} = \lambda^*, \frac{\partial V}{\partial p} = -\lambda^* x^*, \frac{\partial V}{\partial q} = -\lambda^* y^*$$

- You can confirm that the above is exactly what you will get if you differentiate the value function directly.

Another Application: Hotelling's Lemma

- Suppose a competitive firm maximizes its profit $\pi = py - wL - rK$ where the output y is given by the production function $y = F(K, L)$.
- Suppose the optimal values of y, K, L are given by $y^*(p, w, r), K^*(p, w, r)$ and $L^*(p, w, r)$. The value function is thus given by

$$V(p, w, r) = \pi^* = pF(K^*, L^*) - wL^* - rK^*$$

- Note that by the Envelope Theorem,

$$\frac{d\pi^*}{dp} = y^*, \frac{d\pi^*}{dw} = -L^* \text{ and } \frac{d\pi^*}{dr} = -K^*$$

- Hence, the derivatives of the profit function give the output-supply and the input-demand functions.